Atlas of Lie groups and representations
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ノートのタイトル 2007/08/22

Fokko du Cloux

started by Jeff Adams in 2002

Goal : do something like "atlas of finite groups"

Make information about semisimple Lie groups, representations available

Goal : write computer program

(Improve the algorithms available)

input any (real reductive) G $\rightsquigarrow$ output unitary dual of G

proved ~1980's finite calculation

(too hard for $SL(3,\mathbb{R})$ by hand)

HOPE (work of Dan Ciubotaru) $\rightsquigarrow$ general discription of unitary duals for G classical ?

SO HOPE computer program could actually run on E_8.

Atlas project : now ~18 mathematicians

Status du Cloux (2002~2005)

program

input

(any $G(\mathbb{R})$)

G = complex reductive

$\rightsquigarrow$

output

structure theory

Cartan subgps, disconnectedness,

real Weyl groups, $K\backslash G/B$ double cosets

blocks of irr. $(\mathfrak{g}, K)$-modules,

regular infinitesimal character

G cpx $\supset K = G^\theta \leftrightarrow$ real group

Defn (slightly simplified)

$\mathcal{S}$ = all irr. $(\mathfrak{g}, K)$-modules, center of env. alg. acts as in trivial rep of G $= \{x\} \leftarrow$ finite set

$x \sim y$ if $\exists$ x subquo of $y \otimes (S^m(\mathfrak{g}))$

adj. rep of G on polyns.

Make into equiv. relation

Equiv classes are BLOCKS

Fix block $\mathcal{B} = \{x\}$, $x \in \mathcal{B}$

$\rightarrow$ STANDARD REPN $I(x)$ "Verma module"

UNIQUE IRR QUOTIENT $J(x)$ "irr. highest weight module"

KNOW characters of std. reps $I(x)$

(Harish-Chandra, ...)

Can write

$$J(y) = \sum_{x \in \mathcal{B}} M(x,y) \, I(x)$$

integer

finite sum: knowing chars of $J(y)$

$\Leftrightarrow$ knowing integer $M(x,y)$

Thm (Kazhdan-Lustig, Beilinson-Bernstein)

There are polynomials $P(x,y) \in$ non-neg. integer coeff.

$$M(x,y) = (-1)^{l(y)-l(x)} P(x,y)(1)$$

l = "length" function on $\mathcal{B}$-expression

Thm (Kazhdan-Lustig, ...)

There is algorithm to compute $P(x,y)$

If you compute all $P(x,y)$

$\rightsquigarrow$ get $M(x,y) = \pm P(x,y)(1)$

$\rightsquigarrow$ character table for repns $J(y)$ in block

atlas "Nov 2005 - Jan 2007"

run program for $E_8(\mathbb{R})$

compute KL polynomials

(Did E_7, all but one block of E_8 in Nov 2005)

Last E_8 calculation   Marc van Leeuwen

(atlas) modified software

BEILINSON - BERNSTEIN

$\mathcal{S}$ = pairs (Z, ξ)

Z = orbit of K on G/B, ξ = K-equiv. local system on Z

- Hasse diagram of K orbits on G/B

N = # orbit

Graph: rank graph by dimension

edge labelled by simple roots

open orbit $\quad Z_{N-1}$

↑ dimension

$Z_{r_0}, \dots, Z_{r_0+r_1-1}$

closed orbits $\quad Z_0, Z_1, \dots, Z_{r_0-1}$

Matsuki-Ochiai for simple α, minimal parabolic P_α

$$\begin{tikzcd}
G/B \arrow[r, "\mathbb{P}^1\text{-bundle}", "\pi_\alpha"'] & G/P_\alpha \\
\text{K-orbit } Z \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, phantom, "\cap"] & \pi_\alpha(Z) \text{ K-orbit} \arrow[dl, bend left] \\
\pi_\alpha^{-1}\pi_\alpha(Z) &
\end{tikzcd}$$

write $\begin{array}{c} Z' \\ |\alpha \\ Z \end{array}$ if $\begin{array}{l} Z' \subseteq \pi_\alpha^{-1}\pi_\alpha(Z) \\ \dim Z' = \dim Z - 1 \end{array}$

E_8: 320 206 nodes

nodes $\approx$ # elts of order 2 in W

$W(E_8) \sim 7 \times 10^8$

elts of order 2 = 199,852

FACT # elts of order 2 in $W = \sum_{\sigma \in \widehat{W}} \dim \sigma$

E_8 big block : diagram has 453,060 nodes

One K-L polynomial for each pair (x, y)

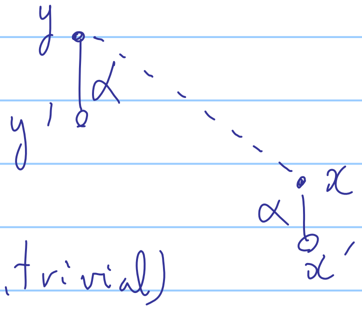
$(0 \leq x, y < 453060)$ Can be non-zero only if

x BELOW y in Hasse diagram.

polyns = # pairs (x, y), x below y

~ 100 billion $\sim 10^{11}$

compute by induction $\uparrow$ on y, then (fixed y) $\downarrow$, x



$(Z_0$, trivial$)$

$\overline{Z}_y = \mathbb{P}^1$-bdle over

$\pi_\alpha(\overline{Z}_y) = \pi_\alpha(\overline{Z}_{y'})$

$\cup$

$\overline{Z}_{y'}$

$\uparrow$

KNOW $P(x', y')$

7/23 ① Finish description of calc. of $E_6(\mathbb{R})$ char table

② Advertise atlas software

③ Unipotent rep. & D-modules

Theme

representation theory — Weyl group repns

geometry of nilpotent elements

$E_8(\mathbb{R})$ block of 453060 repns. numbered

$0, 1, 2, \ldots, 453059$

calculate polyn $P_{x,y}$ $0 \leq x \leq y \leq 453060$ ~ 100 billion pairs

Kazdhan-Lustig geometry : sometimes $P_{x,y} = P_{x',y'}$

related $\begin{matrix} y \\ \mid \alpha \\ y' \end{matrix}$ Matsuki

Pair (x,y) is called PRIMTIVE if there is no "obvious"

KL equality $P_{x,y} = P_{x',y'}$ y' smaller than y

E_8 : # primitive pairs ~ 6 billion (6×10^9)

Computation showed : half give non-zero KL polys (~ 3 billion)

distinct polys ~ 1.1 billion

Partial calc. for $PGL(14, \mathbb{R})$

Found ~ 2 billion non-zero KL polys

~ 1 million distinct

Store : 1.1 billion polyns

coeff $\leq 11{,}808{,}880 < 2^{24}$ $\boxed{3 \text{ bytes}}$

coeff ~ 13 billion $\rightsquigarrow$ $\underset{\sim}{40G}$

For each primitive pair (x,y), store number of KL polyn

on list $0, 1, \ldots, 1.1$ billion $\leftarrow$ 4 byte $< 2^{32}$

$\underset{\sim}{\sim 12G}$ $\Rightarrow$ 50 G disk file

Any $G(\mathbb{R})$, block $\mathcal{B}$ of repns

$\rightsquigarrow$ free $\mathbb{Z}$-module, basis irr reps. $\mathbb{Z}[\mathcal{B}]$

Prop There's natural repn of $W_{\text{cpx Weyl gp}}$ on $\mathbb{Z}[\mathcal{B}]$ α simple root

x : irred repn $\longleftrightarrow$ $1 \cdot x \in \mathbb{Z}[\mathcal{B}]$

$$s_\alpha \cdot x = \begin{cases} -x \\ x + \sum_{y \in \mathcal{B}} \mu(x,y)\, y \end{cases}$$

integer $\mu(x,y) \geq 0$

$\mu(x,y) > 0 \Rightarrow s_\alpha y = x$

Defn $\tau(x) = \{\alpha : \text{simple} \mid s_\alpha x = -x\}$

$\tau : (\text{irr. repns}) \to (\text{subsets of simple roots})$

(Borho - Jantzen - Duflo)

τ easy to compute in "graph" from Matsuki lecture

Roughly : $\tau(x) \longleftrightarrow$ simple roots α for which α edge go DOWN from x

↪ smaller orbit

$\Leftrightarrow$ orbit closure is a $\mathbb{P}^1$ bdle using π_α

UNDERSTANDING IRR REPNS [local system is pullback]

$\longleftrightarrow$ COMPUTING μ

Thm (Part of KL conj)

If $x < y$, $\mu(x,y) =$ coeff of $q^{[\ell(y)-\ell(x)-1]/2}$ in $P_{x,y}$

↖ highest possible term

(KL algorithm ... recursion formula for $\mu(x,y)$ involves lower terms in othe K-L polyn.)

Make W-graph

- Vertices : elements of $\mathcal{B} = \{x\}$
- each vertex labelled by subset of simple roots $\tau(x)$
- edge from x to y, multiplicity $\mu(x,y)$ $\boxed{\text{IF } \tau(y) \subseteq \tau(x)}$

$\boxed{\text{DIRECTED}}$

If $\tau(y) \not\subseteq \tau(x)$ and $\tau(x) \not\subseteq \tau(y)$, edge is bidirected

(Then $\mu(x,y) = 1$; these edge easy to describe by inspection of Matsuki graph.)

HARD : one-direction edges $\tau(y) \supsetneq \tau(x)$

directed graph $\mathcal{B}$ is connected NOT "strongly connected"

cell $\overset{\text{def}}{=}$ strongly connected W-graph

MEANS: have directed path from x to y, any $x, y \in$ cell

Each cell inherits W-graph structure, so W repn def$/\mathbb{Z}$

irr. rep $x \rightsquigarrow$ cell C_x finite set of irr. rep.s $\rightsquigarrow$ W repn (other reducible)

Geometry: x irr. $(\mathfrak{g}, K)$-module

$\rightsquigarrow \mathcal{V}(x) \subset$ nilp cone in $(\mathfrak{g}/\mathfrak{k})^*$

K-inv. closed cone

(associated variety)

Prop $C_x = \{$ irrs y in $\mathcal{B}$ $\mid$ $y \subset x \otimes S(\mathfrak{g})$ $\}$

actual subrep

Cor Assoc variety $\mathcal{V}(\ )$ constant on cell

E_8: $|\mathcal{B}| = 453{,}060$ # cells $= 104$

written down by atlas software

abstract thm:

① $\widehat{W}$ are partitions into "cells". Each cell has exactly one special repn defined by Lustig-Spaltenstein

S_n: all repn. special

(simple) rank 2 triv, sgn, reflection rep. are simple

$1^2 + 1^2 + 2^2 = 6$

Write cell $\mathcal{C} \subset \mathcal{B}$ as $\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{\tau \in \widehat{W}} m_\tau \tau$

All τ's appearing are in one $\hat{W}$-cell ; special repn must appears

numerically $\left(\begin{matrix}\text{cardinality of} \\ \text{any cell}\end{matrix}\right) = \left(\begin{matrix}\text{sum of dims of } W \text{ repns} \\ \text{in one } \hat{W}\text{-cell}\end{matrix}\right)$

$\mathcal{C} \rightsquigarrow$ assoc variety ?

$\downarrow$ special irr rep of $W \xrightarrow[\text{Springer corr.}]{}$ special nilp class in $\mathfrak{g}^* \rightsquigarrow \bar{\mathcal{O}}$

Thm associated variety = union of some of irr comps of $\bar{\mathcal{O}} \cap (\mathfrak{g}/\mathfrak{k})^*$

E_8 : # irr comp ≤ 4 ?