Heckman-Opdam hyperge
functions and their specia

Nobukazu Shimeno (示野 信

Department of Applied Mathema
Okayama University of Scienc

August 26, 2007

Tambara Workshop 2007
Geometry and Representations in Lie

joint work in progress with Toshio Osh

Heckman-Opdam hypergeometric sy
tem) and hypergeometric functions (H

- $\mathfrak{a} \simeq \mathbb{R}^n$ with $\langle\ ,\ \rangle$, $\Sigma \subset \mathfrak{a}^*$ root system of rank n, k_α

  $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$
- HO system $D\,F = \gamma(D)(\lambda)\,F\ (D \in \mathbb{D}(k))$

  $L(k) = \sum_{i=1}^n \partial_i^2 + \sum_{\alpha\in\Sigma^+} 2k_\alpha \coth\alpha\,\partial_\alpha$

  $\mathbb{D}(k)$: commuting family of differential operators $\ni$
- HO HGF $F(\lambda, k; a)$ is a unique analytic solution

  system with $F(\lambda, k; e) = 1$, if k is generic.
- rank one case : $n = 1 \Rightarrow$ HO HGF can be written b
- If $2k_\alpha$ = multiplicity of $\alpha \in \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, then H

  cal functions on a Riemannian symmetric space of

  G/K and $L(k)$, $\mathbb{D}(k)$ is the radial part of the Laplace

  invariant differential operators, respectively. (group

Specializations of HO HGF & HO sys

- Confluences
  - degenerate limit to the class-one Whittaker fun
    ple Lie groups (eigenfunction for Toda model)
  - other Toda-like limits
- Restrictions to 1-dimensional singular sets (interse
  Weyl group)
  - ODE (in some cases without accesory paramet
  - application: value of the HO HGF at the origin
- Real forms
  - a generalization of K-invariant eigenfunctions
    operators on pseudo-Riemmanian symmetric s
  - construction of a basis for the analytic solution

Confluences (1): review of known resu

$$L(k) = \sum_{i=1}^{n} \partial_i^2 + \sum_{\alpha\in\Sigma^+} 2k_\alpha \coth\alpha\,\partial_\alpha\,, \qquad \rho(k) = \sum_{\alpha\in\Sigma^+} k_\alpha \alpha$$

$$L(k)u = (\langle\lambda,\lambda\rangle - \langle\rho(k),\rho(k)\rangle)u$$

Replacing $x = \log a \;\to\; \varepsilon x,\ \lambda \;\to\; \lambda/\varepsilon$ and letting $\varepsilon \downarrow$

becomes

$$L(\tilde{k})_{\mathrm{rat}}\, u = \langle\lambda,\lambda\rangle\, u,$$

where

$$L(\tilde{k})_{\mathrm{rat}} = \sum_{i=1}^{n} \partial_i^2 + \sum_{\alpha\in\Sigma_0^+} \frac{2k_\alpha + 2k_{\alpha/2}}{\alpha}\partial_\alpha$$

$\Sigma_0 = \{\alpha \in \Sigma \,:\, 2\alpha \notin \Sigma\}$, $\tilde{k}_\alpha = k_\alpha + k_{\alpha/2}$ for $\alpha \in \Sigma_0$, $k_{\alpha/2} = 0$

limit transition of W-invariant analytic soluti

Theorem (Ben Saïd and Ørsted, de Jeu)

$$F(\lambda/\varepsilon, k; \exp \varepsilon x) \to J(\lambda, \tilde{k}; \exp x) \quad (\varepsilon$$

where J is the Bessel function associated with Σ_0, whi eigenfunction of commuting family of differential operato

- Special cases (group case and rank 1 case)
  - In the group case, $L(\tilde{k})_{\mathrm{rat}}$ is radial part of the I ator on G_0/K (the tangent space of G/K at the theorem gives a limit transition of the spherica G_0/K.
  - If $n = 1$, then the limit transition in the above t of the Gauss HGF to the Bessel J function.

Confluences (2): limit to the class-
function on real semisimple Lie group

The following content overlaps with my talk in the workshop

Put $\delta(k)^{1/2} = \prod_{\alpha\in\Sigma^+}(2\sinh\alpha)^{k_\alpha}$. Then

$$\delta(k)^{1/2} \circ (L(k) + \langle\rho(k), \rho(k)\rangle) \circ \delta(k)^{-1}$$
$$= \sum_{i=1}^{n} \partial_i^2 + \sum_{\alpha\in\Sigma^+} \frac{k_\alpha(1 - k_\alpha - 2k_{2\alpha})\langle\alpha}{\sinh^2\alpha}$$

The existence of the commuting family of differentia $H_{\mathrm{CMS}}(k)$ proves integrability of the quantum Calogero-Mo model with Hamiltonian $H_{\mathrm{CMS}}(k)$.

limit transition from CMS to Toda Hamiltoni

Temporary, we assume that Σ is reduced. For $M > 0$, de

$$2k_M(\alpha)(k_M(\alpha) - 1)\langle\alpha, \alpha\rangle = e^{2M}$$

and define $a_M \in A$ by

$$\log a_M = w_0 \log a + M\rho^\vee,$$

where $\rho^\vee = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha^\vee$ is the Weyl vector of $\Sigma^\vee = \{\alpha^\vee =$
$w_0 \in W$ is the longest element of the Weyl group W of Σ.
simple roots in Σ^+ and define

$$H_{\mathrm{T}} = \sum_{i=1}^{n} \partial_i^2 - 2\sum_{\alpha\in\Psi} e^{2\alpha}.$$

Lemma (Inozemtsev) For any $\varphi \in C^\infty(A)$,

$$\lim_{M\to\infty} H_{\mathrm{CMS}}(k_M)\,\varphi(a_M) = H_{\mathrm{T}}\,\varphi(a).$$

limit transition of joint eigenfunctions for CM

- limit transition of the Harish-Chandra series

  $\Phi(\lambda, k; a) = a^{\lambda-\rho(k)} + \cdots$: series solution of $L(k)u =$

  $\Phi_{\mathrm{T}}(a) = a^{\lambda} + \cdots$: series solution of $H_{\mathrm{T}}u = \langle\lambda, \lambda\rangle u$

  $\Phi(\lambda, k; a)$ (resp. $\Phi_{\mathrm{T}}(a)$) becomes a joint eigenfuncti ily of differential operators $\mathbb{D}(k)$ (resp. $\mathbb{D}_{\mathrm{T}}$).

$$\delta(k_M; a_M)^{1/2}\Phi(\lambda, k_M; a_M) \to \Phi_{\mathrm{T}}(a) \quad ($$

- definition of HO HGF in terms of the HC series (r reduced)

$$\tilde{c}(\lambda, k) = \prod_{\alpha\in\Sigma^+} \frac{\Gamma((\langle\lambda, \alpha^\vee\rangle + k_{\alpha/2})/2)}{\Gamma((\langle\lambda, \alpha^\vee\rangle + k_{\alpha/2} + 2k_\alpha)/2)}, \quad c($$

$$F(\lambda, k; a) = \sum_{w\in W} c(w\lambda, k)\Phi(w\lambda, k$$

- Remarks
  - In the group case, $c(\lambda, k)$ is Harish-Chandra's c
  - In a series of papers around 1990, Heckman a
    for generic k, $F(\lambda, k; a)$ defined by the above fo
    tinued to $(\lambda, a) \in \mathfrak{a}^*_{\mathbb{C}} \times A$, $F(\lambda, k; e) = 1$, an
    invariant analytic solution of HO system subje
- Problem

What is the limit of $F(\lambda, k; a)$ corresponding to th

$H_{\text{CMS}} \to H_{\text{T}}$ $(M \to \infty,\ k_M \to \infty)$? (Inspired by Hira

- Answer

radial part of the class-one Whittaker function with

real split semisimple Lie group with the restricted r

class-one Whittaker function with moderate g

G : real split semisimple Lie group of finite center,

$G = NAK$: Iwasawa decomposition, $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, $\rho = \frac{1}{2} \sum_{\alpha}$

Define a character ψ of $\mathfrak{n}$ by $\psi(X_\alpha) = \sqrt{-1}$, where $X_\alpha \in \mathfrak{g}^{\alpha}$

length ($\alpha \in \Psi$). $1_\lambda(nak) = a^{\lambda+\rho} \quad (n \in N,\ a \in A,\ k \in K)$

$$W(\lambda, \psi; g) = \int_N 1_\lambda(\overline{w_0}^{-1} ng)\psi(n)^{-1} dn$$

Theorem $\lim_{M\to\infty} \delta(k; a_M)^{1/2}\, \tilde{c}(\rho(k_M), k_M) \prod_{\alpha\in\Sigma^+} \Gamma(k_M(\alpha))\, F(\lambda$

$= \tilde{\boldsymbol{c}}(\rho)\, f(\lambda)\, a^{-\rho}\, W(\lambda, \psi;$

where $\tilde{\boldsymbol{c}}(\lambda)$ is $\tilde{c}(\lambda, k)$ with $k_\alpha = 1/2$ ($\alpha \in \Sigma$) and

$$f(\lambda) = \prod_{\alpha\in\Sigma^+} (2\langle\alpha, \alpha\rangle)^{\langle\lambda,\alpha^\vee\rangle/4}\, \Gamma((\langle\lambda, \alpha^\vee\rangle + $$

- Previously I proved the above theorem by using an
  $W(\lambda, \psi; a)$ in terms of $\Phi_{\mathrm{T}}(w\lambda, a)$ $(w \in W)$ that is due
- In the rank one case, the above theorem is a conflue
  to the Macdonald K function. Define $k_M > 0$ by $4k_{M}$
  we have

$$F(\lambda, k; a_x) = {}_2F_1(\frac{1}{2}(k - \lambda), \frac{1}{2}(k + \lambda)\,;\, k + \cdots$$

$$\lim_{M \to \infty} k_M^{-1/2}\, 2^{-k_M} \sinh^{k_M}(-x + M) F(\lambda, k_M; a_{-x+M} \cdots$$

Confluences (3): limit to Toda-like sys

- If Σ is an irreducible root system with two different
  Toda-like Hamiltonians that are different from H_{T}.

  Toda-BC_n $R(x) = C_0 \sum_{i=1}^{n} e^{-2(x_i - x_{i+1})} + C_3 e^{-2x_n} + C$

  Trig-A_{n-1}-bry-reg

  $R(x) = C_0 \sum_{1\le i<j\le n} \sinh^{-2}(x_i - x_j) + \sum_{l=1}^{n}(C_1 e^{-2x_l} +$

  (Here $R(x)$ is the potential function for the Schrödi

  - Toda-BC_n with $C_0, C_3, C_4 \neq 0$ appears as a ra
    operator with respect to $G = NAK$ with one-d
    tions of N and K for G/K Hermitian symmetri

- Completely integrable systems and their hierarchy
  cal root systems were thoroughly studied by Oshi
  Among quantum integrable systems, CMS model (
  limits form a class whose joint eigenfunctions are e

Theorem 1) (existence of limit as integrable systems) For

and $k_M(\alpha)$ (explicitly given corresponding to the potential

$x \in \mathfrak{a}$ with $x + \upsilon M$ and consider limit as $M \to \infty$. Then the

holomorphically to the confluent commuting system of dif

2) limit of HO HGF is of moderate growth) A suitably

converges to the solution $\overline{W}(x)$ of the confluent system

that is

$$^{\exists}C > 0,\ m > 0 \text{ s.t } |\overline{W}(x)| \le Ce^{m|x|}$$

3) (uniqueness) Global analytic solutions of the confluent

erate growth are unique up to constant multiples.

4) (good estimate for reduced root sytems) If Σ is reduce

estimate of $\overline{W}(x)$ for Toda-Σ (Toda model with the Hamilt

$$|\overline{W}(x)| \le e^{\mathrm{Re}\langle\lambda,x\rangle}, \quad |\overline{W}(x)| \le C \exp(-e^{K\,\mathrm{di}}$$

where C is the open positive Weyl chamber.

Restrictions of HO system to singular

$L(k) = \sum_{i=1}^{n} \partial_i^2 + \sum_{\alpha\in\Sigma^+} 2k_\alpha \coth\alpha\,\partial_\alpha$ and $D \in \mathbb{D}(k)$ h

$\alpha(x) = 0$ for $\alpha \in \Sigma$.

Problem Study ODE satisfied by the restrictions of local ar

sytem on a 1-dimensional singular set.

Case of A_{n-1}

$$L(k) = \sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{1\le i<j\le n} k\coth(x_i - x_j)(\partial_{x_i} - \partial_{x_j})$$

- The restriction of HO system to singular set x_2 =
  (A_{n-1}, A_{n-2})) become ODE of rank n satisfied by ge
  - rank 2 case : calculate the induced DE on the
    computer algebra system Maple.

$z = e^{2(x_1 - x_2)}$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 + \lambda_2 + \lambda_3$

Riemann scheme

$$\left\{ \begin{array}{ccc} z = 0 & z = 1 & z = \infty \\ k + \lambda_1/2 & 0 & k - \lambda_1 \\ k + \lambda_2/2 & 1 - 3k & k - \lambda_2 \\ k + \lambda_3/2 & 2 - 3k & k - \lambda_3 \end{array} \right.$$

local monodromy type $(1, 1, 1)$, $(1, 2)$, $(1, 1, 1)$

This Fuchsian equation is determined by the R

the equation is accessory parameter free.

- general case: Calculating monodromy at the or
  coordinate) by using representations of Hecke
  that local monodromy types are $(1, \ldots, 1)$, $(1,$
  the equation becomes the generalized hyperge

Application: value of HO HGF at the origin

- Proof of $F(\lambda, k; e) = 1$ due to Opdam is indirect. C
  one of motivations to study restrictions of the HO s
- In the case of A_{n-1} we can calculate the value of HG
  using
  - connection formula of HO HGF (c-function)
  - connection formula of GHGF (Okubo-Takano-
  - the following identity for trigonometric functio

$$\sum_{j=1}^{n} \frac{\prod_{1\le i\le n,\, i\ne j} \sin(\frac{1}{2}(\lambda_i - \lambda_j) + k)\pi}{\prod_{1\le i\le n,\, i\ne j} \sin\frac{1}{2}(\lambda_i - \lambda_j)\pi}$$

Case of BC_n

$$L(k) = \sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{l=1}^{n}(k_1 \coth x_l + 2k_2 \coth 2x_l)\partial_{x_l}$$
$$+ \sum_{1\leq i<j\leq n} k_3(\coth(x_i - x_j)(\partial_{x_i} - \partial_{x_j}) + k_3 \coth(x_i$$

- The restriction of HO system to singular set $x_2 = x$ $(B_n, B_{n-1}))$ becomes a Fuchsian DE of rank $2n$ wit points (say $0,\ 1,\ \infty$) on $\mathbb{P}^1$ of local monodromy typ $(n, n),\ (n, n-1, 1),\ (1, \ldots, 1)$, which is free from a (even family EF_{2n} of Simpson (1992)).
  - rank 2 case : by using Maple.
  - general case: by using representations of Heck
- We computed restrictions of HO system also for r $(x_1 = x_2$ for BC_2, $\alpha_1^{\perp}$ or $\alpha_2^{\perp}$ for $G_2)$ by using Maple. there exist accessory parameters.

Real forms of HO system (HO$_\epsilon$ system

A_1 case

$L(k) = \dfrac{d^2}{dx^2} + 2k \coth x \dfrac{d}{dx}$

$F(\lambda, k; x) = c(\lambda, k)\Phi(\lambda, k; x) + c(-\lambda, k)\Phi(-\lambda, k; x)$

$\Phi(\lambda, k; x) = e^{(\lambda-k)x} {}_2F_1(-\lambda + k, k, -\lambda + 1; e^{-2x})$

$k = 1/2 \rightsquigarrow G/K = SL(2, \mathbb{R})/SO(2)$ $(G = KAK)$

$$L(k)_\epsilon = \frac{d^2}{dx^2} + 2k \tanh x \frac{d}{dx} \quad (L(k) \text{ with } x \mapsto x$$

$\Phi_\epsilon(\lambda, k; x) = c(\lambda, k)\Phi(\lambda, k; x + \frac{1}{2}\pi\sqrt{-1})$: analytic eigenfu

$$\Phi_\epsilon(-\lambda, k; -x) = \frac{\sin \pi\lambda}{\sin \pi(\lambda + k)}\Phi_\epsilon(\lambda, k; x) + \frac{\sin \pi k}{\sin \pi(\lambda +}$$

$k = 1/2 \rightsquigarrow G/K_\epsilon = SL(2, \mathbb{R})/SO_0(1, 1)$ $(G = KAK_\epsilon)$

Problem Generalization to higer rank cases

Group case : Oshima-Sekiguchi (1980)

Rank two cases : Sekiguchi (2005)

$\epsilon : \Sigma \to \{\pm 1\}, \quad \epsilon(\alpha+\beta) = \epsilon(\alpha)\epsilon(\beta)$ signature of roots

$$L(k)_\epsilon = \sum_{i=1}^{n} \partial_i^2 + \sum_{\alpha\in\Sigma^+, \epsilon(\alpha)=1} 2k_\alpha \coth\alpha\,\partial_\alpha + \sum_{\alpha\in\Sigma^+, \epsilon(\alpha)}$$

$(L(k)$ with a change of variable $x \mapsto x + \sqrt{-1}v_\epsilon$ for a $v_\epsilon \in$

$\mathbb{D}(k)_\epsilon$: commuting family of differential operators $\ni L(k)_\epsilon$

Group case : $L(k)_\epsilon$ is radial part of the Casimir operator o

(ex. $G/K = SL(n,\mathbb{R})/SO(n)$, $G/K_\epsilon = SL(n,\mathbb{R})/SO_0(p, n$

Radial parts of the Casimir operator on general semisimpl

G/H $(G = KAH)$ are of the form $L(k)_\epsilon$ (Heckman (1994),

$W_\epsilon = \langle s_\alpha \,:\, \epsilon(\alpha) = 1 \rangle \subset W$

$\# W/W_\epsilon = r$, $W_\epsilon \backslash W = \{v_1 = e,\, v_2, \ldots, v_r\}$

$C \subset \mathfrak{a}$: open positive Weyl chamber

Theorem 1) The dimension of the analytic solutions of t

generic k.

2) There exists a basis $F_\epsilon(\lambda, k; x) = (F_\epsilon^{(1)}(\lambda, k; x), \ldots, F$

solutions of HO_ϵ system such that

$$F_\epsilon(\lambda, k; v_i x) = \sum_{w \in W} c(w\lambda, k) A_w^\epsilon(\lambda, k)_{i\text{-th row}} \, \Phi_{v_i}(w\lambda, k, x)$$

Here $\Phi_v(w\lambda, k, x) \sim e^{(\lambda - \rho(k))(x)} + \cdots$ is a series solution on v

are intertwining matrices of size r that satisfy

$$A^\epsilon_{w_1 w_2}(\lambda, k) = A^\epsilon_{w_1}(w_2\lambda, k) A^\epsilon_{w_2}(\lambda, k) \qquad (w_1,\, w$$

$$F_\epsilon(\lambda, k; x) = F_\epsilon(w\lambda, k; x) A^\epsilon_w(\lambda, k) \qquad (w \in$$

For a simple reflection s_α, $A^\epsilon_{s_\alpha}(\lambda, k)$ is a direct product o

of the form

$$A(s,k) = \begin{pmatrix} \frac{\sin\pi k}{\sin\pi(s+k)} & \frac{\sin\pi s}{\sin\pi(s+k)} \\ \frac{\sin\pi s}{\sin\pi(s+k)} & \frac{\sin\pi k}{\sin\pi(s+k)} \end{pmatrix}, \quad \frac{\cos\frac{1}{2}\pi(s}{\cos\frac{1}{2}\pi(s}$$

▼ Example $A_2 : \Psi = \{e_1 - e_2,\ e_2 - e_3\}$

$\epsilon(e_1 - e_2) = 1$, $\epsilon(e_2 - e_3) = -1$, $W_\epsilon = \{1,\ s_1\}$

$$A^\epsilon_{s_1}(\lambda) = \begin{pmatrix} 1 & \\ & A(\lambda_1 - \lambda_2, k) \end{pmatrix}, \quad A^\epsilon_{s_2}(\lambda) = \Bigg(\ A$$

▼ Proof

- rank 1 reduction
- $A_w(\lambda)^\epsilon$ is well-defined (does not depend on a c

  w in terms of simple reflections $\leftsquigarrow$ enough to ch