Observation of (a part of) rainbow that has disappeared

西山

$W_{\mathbb{R}} = \mathbb{R}^{2N}$ simplectic sp.

$Sp(W_{\mathbb{R}}) \supset G, G'$: dual pair

$G' = \{ g' \in Sp(W_{\mathbb{R}}) \mid g'g = gg' \ (\forall g \in G) \}$

$G = \{ g \in Sp(W_{\mathbb{R}}) \mid g'g = gg' \ (\forall g' \in G') \}$

Ex $G = Sp(W_{\mathbb{R}}) \qquad W_{\mathbb{R}} = \mathbb{R}^{2n}$

$G' = O(V^{p,q}), \qquad V^{p,q} = \mathbb{R}^{p,q}$

$\mathcal{W}_{\mathbb{R}} = W_{\mathbb{R}} \otimes V^{p,q}$: symplectic

$G = Sp(W_{\mathbb{R}}) \otimes 1 \hookrightarrow Sp(\mathcal{W}_{\mathbb{R}})$

$G' = 1 \otimes O(V^{p,q}) \hookrightarrow$ 〃 〃

<u>moment map</u> $\mathcal{W}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}^* \qquad \mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}\, Sp(\mathcal{W}_{\mathbb{R}})$

$$w \mapsto \Big(\underset{\in \mathfrak{g}_{\mathbb{R}}}{x \mapsto \tfrac{1}{2}\langle xw, w\rangle} \Big)$$

$\mathbb{C}$-fy

$\Longrightarrow \mathcal{W} = \mathcal{W}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\ \mu\ } \mathfrak{g}^*$ m.m.

$\mathrm{Im}\, \mu =$ min. nilp. orbit

$\mathcal{W} = \mathcal{X} \oplus \mathcal{Y}$: polar decomp

$\mathbb{K} \subset \mathbb{G}$: a max cpt $\simeq U(N)$

$\mathcal{X}$ is stable under $\mathbb{K}$ min nilp $K_{\mathbb{C}}$- orbit

$\mu|_{\mathcal{X}} : \mathcal{X} \longrightarrow \overline{\mathcal{O}_{\min}} \subset \mathfrak{s}^*$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$: Cartan decomp. $/\mathbb{C}$, $\mathcal{X} \ni v$

$\mathcal{X} \ni \begin{pmatrix} v \\ 0 \end{pmatrix} \longmapsto {}^t v\, v \in \mathrm{Sym}_N$

$/\mathbb{Z}_2 = \{\pm 1\}$

$G' \supset K'$ a max cpt s.t.

$K \times K' \subset \mathbb{K} \curvearrowright \mathcal{X}$

$\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ Cartan decomp

$\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S} \Rightarrow \mathfrak{s}^* \to (\mathfrak{s}')^*$

$$\begin{array}{ccc} & \mathcal{X} & \\ & \downarrow \mu & \\ & \overline{\mathcal{O}_{\min}} & \\ \swarrow & & \searrow \\ \mathfrak{s}^* & & \mathfrak{s}'^* \end{array}$$

Ex $G = Sp_{2n}(\mathbb{R})$, $G' = O(p,q)$

$$\mathfrak{X} = M_{n,p} \oplus M_{n,q} \ni (A, B)$$

$$\swarrow \qquad \searrow \; {}^tB A \in M_{q,p}$$

$$(A\,{}^tA,\ B\,{}^tB)$$

$$\mathrm{Sym}_n \oplus \mathrm{Sym}_n^{+} \qquad \mathfrak{g} = \begin{pmatrix} \mathfrak{gl}_n & \mathrm{Sym}_n \\ \mathrm{Sym}_n^{+} & -{}^t\mathfrak{gl}_n \end{pmatrix} \quad \mathfrak{g}' = \begin{pmatrix} \mathrm{Alt}_p & -{}^tM_{q,p} \\ M_{q,p} & \mathrm{Alt}_q \end{pmatrix}$$

<u>Thm</u> (D-K-P, Ohta, N-Ochiai-Zhu)

Assume (G, G') is in the stable range (Ex. $n \geq p+q$)

① ψ : surjective, flat

② $\mathcal{O}' \subset \mathcal{S}'$: $K'_{\mathbb{C}}$-orbit

$\varphi(\psi^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}$ $\quad \exists\, \mathcal{O} \subset \mathcal{S}$ $\quad K_{\mathbb{C}}$-orbit

③ corr. $\mathcal{S}'/K'_{\mathbb{C}} \to \mathcal{S}/K_{\mathbb{C}}$: injective

respecting closure ordering and nilpotency

dual pair correspondence of representations

Ω: Weil repr of $Mp(W_{\mathbb{R}}) = \widetilde{G} \xrightarrow{2:1} Sp(W_{\mathbb{R}}) = G$

$\widetilde{G}, \widetilde{G}'$: inverse image of G, G'

<u>Remark</u> $\mathcal{AV}(\Omega) = \overline{\mathcal{O}_{min}}$

$\Omega|_{\widetilde{G} \times \widetilde{G}'}$, $\quad \widetilde{G}'_{adm} = \{\pi' : \text{irred adm repr of } \widetilde{G}'\}$

$\pi \in \widetilde{G}_{adm}$, $\quad \pi' \in \widetilde{G}'_{adm}$

If $\exists\, \Omega \twoheadrightarrow \pi \otimes \pi'$: $\widetilde{G} \times \widetilde{G}'$-intertwiner,

then π, π' are said to be in dual pair correspondence

Write $\pi = \theta(\pi')$ or $\pi' = \theta(\pi)$

<u>Thm</u> (Howe) the above corr. is one-to-one.

idea of proof

$\pi' \in \widehat{G'}_{adm}$ $\qquad \mathcal{H} = \mathrm{Hom}_{\widetilde{G}'}(\Omega, \pi')$

$$\Omega / \bigcap_{\Phi \in \mathcal{H}} \ker \Phi = \underbrace{\Omega(\pi')}_{\widetilde{G}\text{-rep}} \otimes \underbrace{\pi'}_{\widetilde{G}'}$$

$\Omega(\pi')$ is f.g. adm representation with an inf. char

$\exists_1$ irred quotient $\cong \theta(\pi')$

$\Omega(\pi')$ is called the max quotient for π'

Fact (G, G') : stable range

$\Rightarrow AV(\Omega(\pi')) = \theta(AV(\pi'))$

$$\begin{array}{ccc} \mathfrak{s}'/K'_{\mathbb{C}} & \xrightarrow{\theta} & \mathfrak{s}/K_{\mathbb{C}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{O}' & \longmapsto & \mathcal{O} \end{array}$$

Thm (N, - Zhu)

G' : Hermitian sym. type

(G, G') : stable range

π' : unitary char. or unitary hT wt module

$\Rightarrow \Omega(\pi')$: irreducible $= \theta(\pi')$

$AC(\theta(\pi')) = \theta(AC(\pi'))$.

§ Equal rank case

$\underset{G}{Sp_{2n}(\mathbb{R})} \times \underset{G'}{O(n, n)}$

Consider the lifting of trivial repr. or trivial orbit

Thm (S. T. Lee – C. B. Zhu)

$\Omega(\mathbb{1}) = \mathrm{Ind}_{\tilde{P}}^{G}(|\det|^{n})$

$\tilde{P} \cong GL_n(\mathbb{R}) \ltimes Sym_n(\mathbb{R})$

$\varphi(\psi^{-1}(\{0\})) \subset S$: $K_{\mathbb{C}}$-stable

$\quad = \bigcup_{p+q=n} \overline{\mathcal{O}_{p,q}} \qquad G_{\mathbb{C}} \cdot \mathcal{O}_{p,q} = [2^n]$

$$\begin{array}{l} \left.\begin{matrix} + & - \\ \vdots & \vdots \\ + & - \end{matrix}\right\} p \\ \left.\begin{matrix} - & + \\ \vdots & \vdots \\ - & + \end{matrix}\right\} q \end{array}$$

Assume n is even

$\Theta(\mathbb{1})$

$$\begin{tikzcd}[column sep=small]
 & & & \pi_{1,1} \arrow[dl] \arrow[dr] & & & \\
 & & \pi_{3,1} \arrow[dl] \arrow[dr] & & \pi_{1,3} \arrow[dl] \arrow[dr] & & \\
 & \pi_{5,1} \arrow[dl] \arrow[dr] & & \pi_{3,3} \arrow[dl] \arrow[dr] & & \pi_{1,5} \arrow[dl] \arrow[dr] & \\
\pi_{6,0} & & \pi_{4,2} & & \pi_{2,4} & & \pi_{6,0}
\end{tikzcd}$$

Hasse diagram of $\mathbb{1}$ deg PS

max GK-dim

unitary

$AV(\pi_{i,j}) = \overline{\mathcal{O}_{i,j}}$

$\mathcal{O}(i,j)$ nilp orbit

$$\begin{array}{l} \left.\begin{matrix} + & - \\ \vdots & \vdots \\ + & - \end{matrix}\right\} i \\ \left.\begin{matrix} - & + \\ \vdots & \vdots \\ - & + \end{matrix}\right\} j \\ \begin{matrix} + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{matrix} \end{array}$$

$\overline{\mathcal{O}_{p,q}} = \overline{\mathcal{O}_p} \cap S$

$G_{\mathbb{C}} \cdot \mathcal{O}_{p,q} = [2^n] = \mathbb{O}_P$

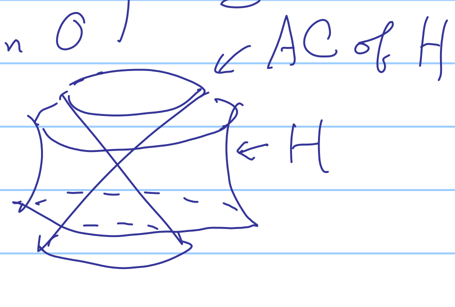
$\mathbb{C}[\overline{\mathbb{O}_P} \cap S]|_{K_{\mathbb{C}}} \simeq \Omega(\mathbb{1})|_K$

ss orbit $\quad G_{\mathbb{R}} \cdot a$

a s.s. element $\qquad a = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in S$

$\underset{\sim}{AC}(K_{\mathbb{C}} \cdot a) = \overline{\mathbb{O}_P} \cap S$

asymptotic cone



$\varphi(\psi^{-1}(\{0\})) = \mathcal{N}$ null cone

$= \{(A,B) \in M_{n,n} \times M_{n,n} \mid {}^tBA = 0\}$

$= \bigcup_{n=p+q} \mathcal{N}_{p,q}$: irred. decomposition

$= \{(A,B) \in \mathcal{N} \mid \operatorname{rank} A = p, \operatorname{rank} B = q\}$

$\mathcal{N}_{p,q} \ni (A,B) \mapsto \operatorname{Im} A = \operatorname{Ker} {}^tB$

q-dim subsp $\subset \mathbb{C}^n$

$\in \operatorname{Grass}_p(\mathbb{C}^n)$

closed $K_{\mathbb{C}}$-orbit $\longrightarrow$ on $G_{\mathbb{C}}/P_{\mathbb{C}}$ = Lagransian Grassmanian

<u>Theorem</u> $\exists\ \tilde{\mathcal{N}}_{p,q} \longrightarrow \mathcal{N}_{p,q}$

$K_{\mathbb{C}} \times K'_{\mathbb{C}}$ - equivariant

resolution of singularities

$\tilde{\mathcal{N}}_{p,q}$: vector space / $\operatorname{Grass}_p(\mathbb{C}^n)$

$\tilde{\mathcal{N}}_{p,q} /\!/ K'_{\mathbb{C}} \simeq T^*_{Z_{p,q}}(G_{\mathbb{C}}/P_{\mathbb{C}}) \xrightarrow{m,m} \mathcal{O}_{p,q}$

conormal bdle

<u>Hope</u> $\pi_{p,q} \longleftrightarrow Z_{p,q}$ + trivial. local system which is the unitary irreducible subrepres. of $\Omega(\mathrm{II})$ deg <u>P.S.</u>

quantization of nilization of nilp orbits

$U(n,n) \times U(n,n)$