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Date 2007 8 · 24

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On homomorphisms between scalar generalized Verma modules

$\mathfrak{g}$: $\mathbb{C}$ ss. $\supset \mathfrak{h}$ CSA Δ : root system $\supset \Delta^+$: pos. system $\supset \Pi$: basis

W : Weyl group. $\mathfrak{b}$: Borel subalg $\mathfrak{b} \supseteq \mathfrak{h} \longleftrightarrow \Delta^+$

$\{ \Theta \mid \Theta \subseteq \Pi \} \longleftrightarrow \{ \text{p.s.alg} \supseteq \mathfrak{b} \}$ Θ : basis of the root system of the Levi part of $\mathfrak{p}_\Theta$

$\Theta \longleftrightarrow \mathfrak{p}_\Theta$

$\mathfrak{a}_\Theta = \{ X \in \mathfrak{h} \mid \beta(X) = 0 \ (\beta \in \Theta) \}$ $\mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ via $\langle \, , \rangle$

$\rho_\Theta = \frac{1}{2} \sum_{\alpha \in \Delta^+ \cap \mathbb{Z}\Theta} \alpha$ $\rho^\Theta = \rho - \rho_\Theta$ $\rho^\Theta \in \mathfrak{a}_\Theta^*$

$\rho + \mathfrak{a}_\Theta^* = \rho_\Theta + \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ $\lambda \in \mathfrak{a}_\Theta^*$ $\mathbb{C}_\lambda$: 1-dim'l $\mathfrak{p}_\Theta$-module

$M_\Theta[\lambda] \overset{\text{def}}{=} U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} \mathbb{C}_{\lambda - \rho^\Theta}$: scalar gen. Verma module

inf. character $= \rho_\Theta + \lambda$

Problem $\lambda, \mu \in \mathfrak{a}_\Theta^*$ $\dim \mathrm{Hom}_{U(\mathfrak{g})}(M_\Theta[\lambda], M_\Theta[\mu]) = ?$

Theorem (Lepowsky) (1) $\dim \mathrm{Hom}_{U(\mathfrak{g})}(M_\Theta[\lambda], M_\Theta[\mu]) \leq 1$

(2) $\varphi \in \mathrm{Hom}_{U(\mathfrak{g})}(M_\Theta[\lambda], M_\Theta[\mu])$ is injective.

Problem When is $M_\Theta[\lambda] \hookrightarrow M_\Theta[\mu]$?

$\exists w \in W$ $w(\rho_\Theta + \lambda) = \rho_\Theta + \mu$. necessary cond.

Solved for $\Theta = \phi \rightarrow \mathfrak{p}_\Theta = \mathfrak{b} \rightarrow$ BGG.

for $\mathfrak{p}_\Theta$: max. psalg.

necessary cond. 1

Def $W(\Theta) = \{ w \in W \mid w\Theta = \Theta \}$: subgroup of W

Remark • $w \in W(\Theta)$ $w\rho_\Theta = \rho_\Theta$

• $W(\Theta)$ preserves $\mathfrak{a}_\Theta^*$

Prop $\lambda, \mu \in \mathfrak{a}_\Theta^*$ $w \in W$ s.t.

(1) $\rho_{\Theta} + \lambda$, $\rho_{\Theta} + \mu$ are integral
(2) $\rho_{\Theta} + \lambda$, $\rho_{\Theta} + \mu$ are regular
(3) $w(\rho_{\Theta} + \lambda) = \rho_{\Theta} + \mu$
(4) $M_{\Theta}[\lambda] \hookrightarrow M_{\Theta}[\mu]$
Then, $w \in W(\Theta)$

Idea of proof
τ-invariant
$\tau(V) \subseteq \Pi$. V: irred $U(\mathfrak{g})$-mod
$\tau(V)$ only depends on $\mathrm{Ann}_{U(\mathfrak{g})}(V)$

$I_1 = \mathrm{Ann}_{U(\mathfrak{g})}(M_{\Theta}[\mu])$
$I_2 = \mathrm{Ann}_{U(\mathfrak{g})}(M_{\Theta}[\lambda])$ } primitive ideal
Using Borho-Jantzen's result, we can easily calculate $\tau(I_1), \tau(I_2) \subset \Pi$
$I_2 \supseteq I_1$ from (4): Considering GK-dim, we have $I_2 = I_1$
So $\tau(I_1) = \tau(I_2) \longrightarrow w\Theta = \Theta$ $\therefore$ $w \in W(\Theta)$ //
In fact, using Soergel's theorem, we can reduce the problem to the integral inf. character setting.

What is $W(\Theta)$?

Lemma $r: W(\Theta) \longrightarrow O(\mathfrak{a}_{\Theta}^*, \langle , \rangle|_{\mathfrak{a}_{\Theta}^*})$ is ~~surjective~~ injective

$w \longmapsto w|_{\mathfrak{a}_{\Theta}^*}$

(!) $w \in W(\Theta) \subseteq W$ s.t. $w|_{\mathfrak{a}_{\Theta}^*} = id$.
$w\rho = w\rho_{\Theta} + w\rho^{\Theta} = \rho_{\Theta} + \rho^{\Theta} = \rho$ $\therefore$ $w = id$ //

Def (Something like restricted root system)
$\Sigma_{\Theta} = \{\alpha|_{\mathfrak{a}_{\Theta}} \mid \alpha \in \Delta\} \setminus \{0\} \subseteq \mathfrak{a}_{\Theta}^*$
↳ In general, it is not (non-reduced) root system.

Def A p.s.alg. $\mathfrak{p}$ of $\mathfrak{g}$ is called normal if for any psalg $\mathfrak{p}'$ of $\mathfrak{g}$ with a common Levi part with $\mathfrak{p}$. $\mathfrak{p}$ and $\mathfrak{p}'$ are conj. under inner automorphism.

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- $\mathfrak{p}_\Theta$ is normal iff $\forall w \in W\ (w\Theta \subseteq \Pi \to w \in W(\Theta))$

Ex If $\mathfrak{p}$ is complexified min psalg for some real form s.t.  does not appear in its Satake diagram. Then $\mathfrak{p}$ is normal.

Classifying normal psalg $\subseteq$ Oshima's lecture

Classical case $(A_{n-1})_k$ $k|n$ $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{C})$

Levi part $\simeq \underbrace{\mathfrak{gl}(k,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k,\mathbb{C})}_{(n/k)\text{-times}}$

$X = B, C, D$ $(X_n)_{k,m}$ $m<n$ $k|(n-m)$

Levi part $\simeq \underbrace{\mathfrak{gl}(k,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k,\mathbb{C})}_{((n-m)/k)\text{-copies}} \oplus X_m$ $X_0 = 0$

except $(D_n)_{k,0}$ k: odd & $k>1$

$(X_n)_{k,m}$'s are all normal & These exhaust the normal psalg /up to inner autom.

Prop (Case-by-Case) If $\mathfrak{p}_\Theta$ is normal, Σ_Θ is a (possibly non-reduced) root system. In this case, we also have $W(\Theta) = W(\Sigma_\Theta)$

Def Fix $\alpha \in \Delta^+$, $\Delta(\alpha) = \{\beta \in \Delta \mid \exists c \in \mathbb{R}\ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\}$

$\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+$, $\Pi(\alpha)$: basis of $\Delta^+(\alpha)$

$\boxed{\Theta \subset \Pi(\alpha)}$ $W(\alpha)$: the Weyl group for $\Delta(\alpha)$. $W(\alpha) \subseteq W$.

$w^\alpha \in W(\alpha)$: the longest element

W_Θ: the Weyl group for $(\Delta \cap \mathbb{Z}(\Theta), \Theta)$

$w_\Theta \in W_\Theta$: the longest element

Def $\alpha \in \Delta^+$ (1) $\sigma_\alpha = w_\Theta w^\alpha$

(2) $\alpha \in \Delta^+$ is called Θ-acceptable if $w^\alpha w_\Theta = w_\Theta w^\alpha \leftrightarrow \sigma_\alpha^2 = 1$

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$S_{\Theta} = \{\sigma_\alpha \mid \alpha \in \Pi - \Theta\}$

Lemma (Case-by-Case) (1) If $\mathfrak{p}_\Theta$ is normal if and only if $\forall \alpha \in (\Pi - \Theta)$ is Θ-acceptable. In this case $S_\Theta \subseteq W(\Theta)$)

(2) Assume $\mathfrak{p}_\Theta$ is normal. Then $(W(\Theta), S_\Theta)$ is a Coxeter system

Def $(W, \leq)$: Bruhat ordering wrt Π

$\mathfrak{p}_\Theta$: normal $(W(\Theta), \leq_\Theta)$: Bruhat ordering wrt S_Θ

Key lemma $\mathfrak{p}_\Theta$: normal. For $x, y \in W(\Theta)$, the following cond.'s are equivalent. (1) $x \leq y$ (2) $x \leq_\Theta y$

Famous fact (Lepowsky?) $\rho_\Theta + \lambda$: dominant reg. integral, $x, y \in W(\Theta)$.

$M_\Theta[x\lambda] \hookrightarrow M_\Theta[y\lambda]$ only if $x \geq y$

Cor $\lambda \in \Omega^*_\Theta$ $\rho_\Theta + \lambda$: dom. reg. int, $x, y \in W(\Theta)$

$\mathfrak{p}_\Theta$: normal $M_\Theta[x\lambda] \subseteq M_\Theta[y\lambda]$ only if $x \geq_\Theta y$

Def (1) $\alpha \in \Delta^+$: Θ-acceptable, α is called Θ-excellent if $M[\sigma_\alpha \lambda] \hookrightarrow M[\lambda]$ for any $\lambda \in \Omega^*_\Theta$ s.t. $\langle \lambda, \alpha^\vee \rangle > 0$ and $\rho_\Theta + \lambda$ is integral

we know when it happens ([M-2006])

(2) $\mathfrak{p}_\Theta$ is called strictly normal if any $\alpha \in \Pi \setminus \Theta$ is Θ-excellent

Theorem $\lambda \in \Omega^*_\Theta$, $\rho_\Theta + \lambda$ is dom. reg. integral, $\mathfrak{p}_\Theta$ is strictly normal then for any $x, y \in W(\Theta)$,

$M_\Theta[x\lambda] \hookrightarrow M_\Theta[y\lambda]$ if and only if $x \geq_\Theta y$

Classical case $(A_{n-1})_k$; $(B_n)_{2k,m}$ $(k \leqq 2m)$, $(B_n)_{2k+1,m}$ $(k \geqq 2m)$, $(C_n)_{2k,m}$ $(k \leqq 2m)$ $(C_n)_{2k+1,m}$ $(k \geqq 2m)$; $(D_n)_{2k-1,m}$ $(k \leq 2m)$, $(D_n)_{2k,m}$ $(k \geqq 2m)$ $(D_n)_{1,0}$

classification of strictly normal parabolics E_8 … White board [illegible]