# Generalized Schubert cells and the complex crown

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## 1. The duality

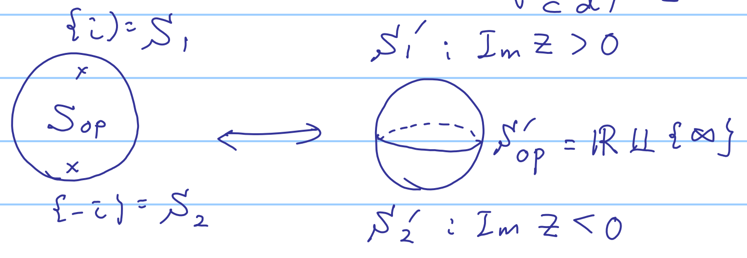
$G_{\mathbb{C}}$ complex s-s Lie group
$\cup$
$G$ connected real form
$\cup$
$K$ a max compact subgroup
$\cap$
$K_{\mathbb{C}}$  $X = G_{\mathbb{C}}/P$ a flat manifold
$\cap$
$G_{\mathbb{C}}$

$$K_{\mathbb{C}}\backslash X \overset{1:1}{\longleftrightarrow} G\backslash X$$
$$S \longleftrightarrow S' \qquad [M79, M82]$$

by the condition $S \cap S'$ is non-empty and compact [M88]
(conj. by J. A. Wolf 1984)

[Ex] $G_{\mathbb{C}} = SL(2,\mathbb{C})$, $G = SL(2,\mathbb{R})$, $K = SO(2)$, $K_{\mathbb{C}} = SO(2,\mathbb{C})$

$X = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \amalg \{\infty\}$ $\quad G_{\mathbb{C}}$ acts on $X$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$




In 1999, S. Gindikin asked me : $x \in G_{\mathbb{C}}$

$C(S) := \{x \in G_{\mathbb{C}} \mid xS \cap S' \text{ is non empty and compact}\}$

What is $C(S)$ and $\bigcap_{S,P} C(S)$ ? $\quad X = G_{\mathbb{C}}/P$

Rem. $C(S)$ is left $G$-inv. and right $K_{\mathbb{C}}$-invariant

$G\backslash G_{\mathbb{C}}/K_{\mathbb{C}}$

Ex 1 (cont) $C(S_1) = \{x \in G_{\mathbb{C}} \mid xS_1 \cap S_1' \neq \phi,\ \text{compact}\}$

$= \{x \in G_{\mathbb{C}} \mid xS_1 \subset S_1'\}$

$C(S_1)/\sim$ is called "cycle space" [Wells-Wolf 77]

$C(S_2) = \{x \in G_{\mathbb{C}} \mid xS_2 \subset S_2'\}$

$C(S_{op}) = \{x \in G_{\mathbb{C}} \mid xS_{op} \supset S_{op}'\}$ (4-conn. components)

$C(S_{op})_0 = \{x \in G_{\mathbb{C}} \mid xS_1 \subset S_1',\ xS_2 \subset S_2'\}$

(= "Iwasawa domain")

$= C(S_1) \cap C(S_2) \underset{\uparrow}{=} D$

In general [GM03, Burs-Halverscheid-Hind03]

$D := G(\exp t^+)K_{\mathbb{C}}$ [Akiezer-Gindikin 90]

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ : Cartan decomp.

$t$ : a max abelian subsp of $i\mathfrak{m}$

$t^+ = \{Y \in t \mid |\alpha(Y)| < \frac{\pi}{2}\ (\forall \alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, t))\}$

Theorem (Conj 1.6 in [GM03]) many people + [M07]

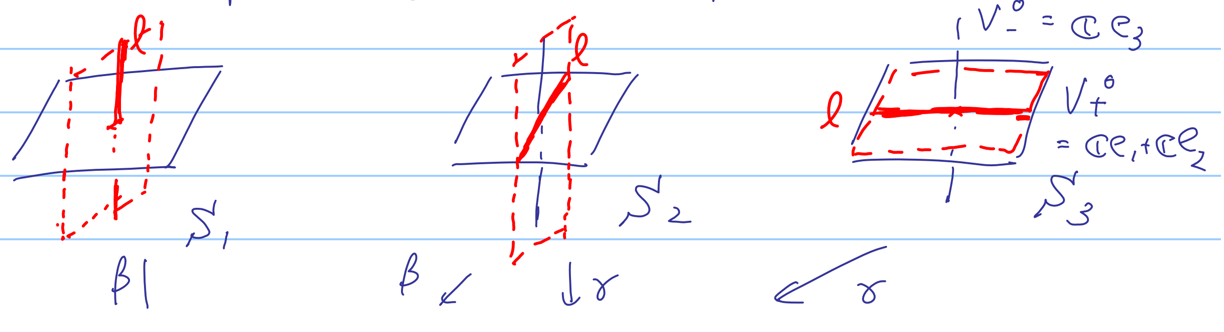
For $\forall K_{\mathbb{C}}$-orbit $S \subsetneq G_{\mathbb{C}}/P$ (for $\forall P$) of nonholomorphic type. Assume $G$ is simple. $C(S')_0 = D$

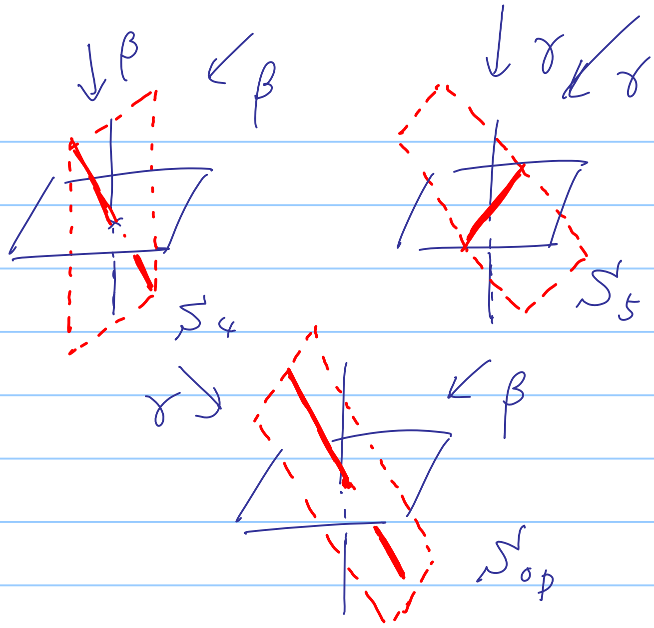
Ex. 2 $G_{\mathbb{C}} = SL(3,\mathbb{C})$, $G = SU(2,1)$,

$K = \left\{\begin{pmatrix} A & \\ & b \end{pmatrix} \;\middle|\; A \in U(2),\ (\det A)b = 1\right\}$

$K_{\mathbb{C}} = \left\{\begin{pmatrix} A & \\ & b \end{pmatrix} \;\middle|\; A \in GL(2,\mathbb{C}),\ (\det A)b = 1\right\}$

$X = \{(\ell, p) \mid \dim_{\mathbb{C}} \ell = 1,\ \dim_{\mathbb{C}} p = 2,\ \ell \subset p\} \simeq G_{\mathbb{C}}/P$






$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$

$P_\beta = \{ \quad \}$

$P_\gamma = \{ \quad \}$

$$\begin{tikzcd} & G_{\mathbb{C}}/B \arrow[dl] \arrow[dr] & \\ G_{\mathbb{C}}/P_\beta & & G_{\mathbb{C}}/P_\gamma \end{tikzcd}$$

$|z_1|^2 + |z_2|^2 - |z_3|^2 = 0$

$$S_4^{cl} = S_1 \sqcup S_3 \sqcup S_4 = S_1 P_\beta = S_3 P_\beta = S_4 P_\beta$$

For $\forall S \in K_{\mathbb{C}} \backslash G_{\mathbb{C}} / B$

we can take $S(0), \dots, S(k) \in K_{\mathbb{C}} \backslash G_{\mathbb{C}} / B$

and $\alpha_1, \dots, \alpha_k \in \Pi$ such that

(i) $S(0)$ is closed (ii) $S = S(k)$

(iii) $S(j) \subset S(j-1) P_{\alpha_j}$ and $\dim_{\mathbb{C}} S(j) = \dim_{\mathbb{C}} S(j-1) + 1$ for $j = 1, \dots, k$

**Prop** [Springer 84] (cf. [Vogan 83])

$$S^{cl} = S(0) P_{\alpha_1} \cdots P_{\alpha_k} = S(0) (BwB)^{cl}$$
$$w = w_{\alpha_1} \cdots w_{\alpha_k}$$

$$G_{\mathbb{C}} = S_{op}^{cl} = S(0) P_{\alpha_1} \cdots P_{\alpha_\ell}$$

For $x \in G_{\mathbb{C}}$, $I_j(x) := x S(j)^{cl} \cap S'_{op} P_{\alpha_\ell} \cdots P_{\alpha_{j+1}}$ $\cap S'(j)$

**Th 1** [M06b] If $I_0(x)$ is connected, then $I_j(x)$ is connected for $j = 1, \dots, \ell$.

· $D \subset C(S)_0$ easier part (many people contribute)
· $D \supset C(S)_0$ harder part
$x_0 S \cap S' \subset x_0 S^{cl} \cap S'^{cl}$ (Th1. implies $\subset \Rightarrow =$)

Lem For $\forall g \in G_{\mathbb{C}}$, every $(g P_\alpha g^{-1} \cap G)_0$-invariant closed subset of $g P_\alpha / B$ is connected.

Conj 1.3 [GM03]

$$\bigcap_{S, P} C(S) = DZ \; ? \qquad Z : \text{center of } G_{\mathbb{C}}$$

· open and nonHermitian case