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Lie algebra cohomology and branching rule

G : semisimple Lie group, connected, finite center. $\mathfrak{g}_0 = \mathrm{Lie}\, G$, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

θ : Cartan involution $K = G^\theta$: a max'l compact subgroup.

R : closed reductive subgroup of G, connected, θ-stable.

1. X^λ : finite dim'l irred. repr. of G, Y^μ : —"— of R.

$\mathrm{Hom}_R(Y^\mu, X^\lambda|_R)$ ... Kostant formula.

2. X_λ discrete series of G with HC parameter $\lambda+\rho$

Y_μ " " R " " $\mu + \rho(\cdot r)$

<u>Fact</u> If $X_\lambda = \bigoplus_{i \in \hat{R}} m_i V_i$ (discretely) $\Longrightarrow$ V_i : discrete series for R.

3. X_λ discrete series for symmetric space G/H.

Assume. $X_\lambda|_R$ decomposes discretely: $X_\lambda|_R = \bigoplus m_i V_i$

<u>Question</u> Is discrete series for $R/R\cap H$

4. $X = A_{\mathfrak{q}}(\lambda)$ unitary

Assume $A_{\mathfrak{q}}(\lambda)|_R$ decomposes discretely $= \bigoplus m_i V_i$

<u>Question</u> Is V_i an $A_{\mathfrak{q}}(\lambda)$-module for R ?

Vogan, "Lie algebra cohomology and Kostant multiplicity formula"

G, R : compact groups $T \subset R$: CSG extend $T \subset H$: CSG for G.

$\mathfrak{t}_0 \subset \mathfrak{h}_0$, $\mathfrak{t} \subset \mathfrak{h}$ $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ $\Delta_{\mathfrak{t}} = \Delta(\mathfrak{g}, \mathfrak{t})$

$\mathrm{Res}: \mathfrak{h}^* \longrightarrow \mathfrak{t}^*$ choose $\Delta^+ \subset \Delta$ s.t. $\overline{\Delta^+} = \Delta^+_{\mathfrak{t}}(\mathfrak{g}, \mathfrak{t})$

(denoted by $\bar{\ }$)

Take a maximal regular element $x \in i\mathfrak{t}_0$ s.t. $\alpha \in \Delta$, $\alpha(x) = 0$

$\Rightarrow \bar{\alpha} = 0$

Define $\mathfrak{q} \supseteq \mathfrak{h}$ to be the parabolic

$\Delta(\mathfrak{q}) = \{\alpha \in \Delta \mid \alpha(x) \geq 0\}$ $\Delta(\mathfrak{l}) = \{\alpha \in \Delta \mid \alpha(x) = 0\}$

$\mathfrak{l} = \mathfrak{g}^{\mathfrak{t}}$ Then $\mathfrak{q} = \underbrace{\mathfrak{q} \cap \mathfrak{r}}_{\text{Borel for } \mathfrak{r}} \oplus \mathfrak{q} \cap \mathfrak{s}$

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Kostant's branching law for f.d. repns

X^λ : irred. repr. of G with h.w. λ

Y^μ : — " — R — " — μ

$$\dim \mathrm{Hom}_R(Y^\mu, X^\lambda) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} d_\sigma(\lambda) \mathcal{P}_{\mathfrak{u}\cap\mathfrak{s}}(\overline{\sigma(\lambda+\rho) - \rho - \mu})$$

where $d_\sigma(\lambda) = $ dim of $\mathfrak{l}$-module with h.w. $\sigma \cdot \lambda = \sigma(\lambda+\rho) - \rho(\mathfrak{l})$

and W^1 is a subset of W $(\in W(\mathfrak{g}, \mathfrak{j}))$ defined as follows

for $\sigma \in W$, $\Delta_\sigma^+ = \{\alpha \in \Delta^+ \mid \sigma^{-1}\alpha < 0\}$ $\quad \ell(\sigma) = |\Delta_\sigma^+|$

$W^1 = \{\sigma \in W \mid \Delta_\sigma^+ \subseteq \Delta(\mathfrak{u})\}$ $\quad W^1 \times W(\mathfrak{l}) \xrightarrow{\sim} W$: bijection

$(\sigma, w) \longmapsto \sigma \cdot w$

idea of proof $\quad X^\lambda = \oplus m_\mu Y^\mu$

consider Lie alg. cohomology $H(\mathfrak{u}, X) \otimes (\dots) \longleftrightarrow H(\mathfrak{u}\cap\mathfrak{r}, X)$

X_λ : d.s. of G with HC parameter $\lambda + \rho$.

$X_\lambda|_R$ is R-admissible $X_\lambda|_R = \oplus m_\mu Y_\mu$ $\quad$ d.s. of R HC parameter μ.

$\Rightarrow X_\lambda|_{K_R}$ is K_R-admissible

Now assume $K' \subseteq K$ and X_π is a Harish-Chandra module

X_π is K'-admissible $\iff AS_K(X_\pi) \cap \mathfrak{k}'^\perp = \{0\}$

Let X_λ be a discrete series with Harish-Chandra parameter $\lambda + \rho$. Assume that $X_\lambda|_{K_R}$ is K_R-admissible.

In addition, assume rank R = rank G (rank K_R = rank K)

Define W_K^1 a subset of W_K so that $W_K = W_K^1 \times W_{K_R}$

$\sigma \in W_K^1 \iff \sigma\rho_c$ is Δ_c^+-dominant.

$X_\lambda = \oplus m_\mu Y_\mu$ $\quad$ Y_μ discrete series of R with H.C. parameter $\mu + \rho(\mathfrak{r})$.

Then $m_\mu = \mathrm{Hom}_R(Y_\mu, X_\lambda)$

$$= \sum_{\sigma \in W_{K_R}} \sum_{w \in W_K^1} (-1)^{\ell(\sigma)} P_s(\sigma w(\mu + \rho(r)) - \rho(s) - (\lambda + \rho)).$$

Here P_s is the partition function of s

$P_s(\xi) = \#\{\text{expressions } \xi = \sum_{\beta \in \Delta^+(s)} n_\beta \beta\}$.

$\Delta^+ = \Delta^+(r) \cup \Delta^+(s)$. (Recall $\mathfrak{g} = r \oplus s$).

In case $R = K$, $r = \mathfrak{k}$, $s = \mathfrak{p}$, $W_{K_R} = W_K$, $W_K^1 = 1$.

The above formula reduces

$$m_\mu = \sum_{\sigma \in W_K} (-1)^{\ell(\sigma)} P_p(\sigma(\mu + \rho_c) - \rho_n - (\lambda + \rho)).$$

The Blattner formula.