# On the packets for inner forms of SL(N)

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Joint work with Hiroshi Saito.

Today: tempered $L$-packets and $A$-packets.

# §1 Introduction

The arguments in this introduction are also contained in
J. Arthur, "A note on $L$-packets."
Pure Appl. Math. Q. 2 (2006), no. 1, 199–217.

$F$: $p$-adic field.

$W_F$: Weil group.

$F$ is a finite extension of $\mathbb{Q}_p$.

$W_F \subset \mathrm{Gal}(\overline{F}/F)$: dense subgroup.

# §1 Introduction

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$F$: $p$-adic field.

$W_F$: Weil group.

$\tilde{G} = GL_m(D)$: inner form of $GL(N)$.

$G = \ker[GL_m(D) \xrightarrow{Reduced\, Norm} F^\times]$: inner form of $SL(N)$.

$L$-parameter:

$\tilde{\phi} : W_F \times SU(2) \longrightarrow GL(N, \mathbb{C}) \times W_F$ or $GL(N, \mathbb{C})$.

$\phi : W_F \times SU(2) \longrightarrow PGL(N, \mathbb{C}) \times W_F$ or $PGL(N, \mathbb{C})$.

$G$: connected reductive algebraic group over the $p$-adic field $F$.

**Conjectural local Langlands correspondence**

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In general, $L$-packet $\Pi_\phi(G)$ is a finite set of irreducible admissible representations of $G$.

**Problem**

How to parametrize the representations in $\Pi_\phi(G)$ ?

In the paper

Labesse, and Langlands, "$L$-indistinguishability for SL(2)". Canad. J. Math. 31 (1979), no. 4, 726–785.

Labesse–Langlands studied this problem by using the endoscopy.

I will explain a typical example in [Labesse–Langlands].

## Typical example in [Labesse–Langlands]

$G = SL(2)$, $\hat{G} = PGL(2, \mathbb{C})$.

Let $K$ be a Galois extension of $F$ such that

$$\mathrm{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

$$\mathrm{Gal}(K/F) = \{1,\ \tau_1,\ \tau_2,\ \tau_1\tau_2\}$$

Define an $L$-parameter $\phi$ by

$$\begin{array}{rcl} \phi : W_F \longrightarrow \ \mathrm{Gal}(K/F) & \longrightarrow & PGL(2,\mathbb{C}) \\ \tau_1 & \longrightarrow & \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix} \\ \tau_2 & \longrightarrow & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \end{array}$$

We choose $\tilde{\phi} : W_F \longrightarrow GL(2,\mathbb{C})$ so that

$$W_F \xrightarrow{\tilde{\phi}} GL(2,\mathbb{C}) \xrightarrow{\text{proj}} PGL(2,\mathbb{C})$$

is equal to $\phi$.

$L$-parameter $\tilde{\phi}$ $\quad\overset{\text{local Langlands}}{\longleftrightarrow}\quad$ $\Pi_{\tilde{\phi}}(GL(2)) = \{\pi(\tilde{\phi})\}$

$\pi(\tilde{\phi})$ is an irreducible supercuspidal representation of $GL(2,F)$.

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We have

$$\mathrm{Res}_{SL(2)}^{GL(2)}\, \pi(\tilde{\phi}) = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4.$$

| | | |
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| $L$-parameter $\phi$ | $\longleftrightarrow$ | $\Pi_{\phi}(SL(2)) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ |

We have

$$C_\phi = \mathrm{Cent}(\phi, PGL(2,\mathbb{C})) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

$\mathrm{Cent}(\phi, PGL(2,\mathbb{C}))$: centralizer of $\mathrm{Im}\,\phi$ in $PGL(2,\mathbb{C})$.

$$C_\phi = \left\langle s_1 = \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}, s_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\rangle.$$

$$\begin{array}{cccc} \phi : W_F \longrightarrow & \mathrm{Gal}(K/F) & \longrightarrow & PGL(2,\mathbb{C}) \\ & \tau_1 & \longrightarrow & \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix} \\ & \tau_2 & \longrightarrow & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \end{array}$$

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$\Pi(C_\phi)$: the set of irreducible representations of $C_\phi$.

$$\Pi_\phi(SL(2)) \quad \overset{1:1}{\longleftrightarrow} \quad \Pi(C_\phi).$$

$\Pi_\phi(SL(2)) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$.

$\sharp\Pi(C_\phi) = 4$.

$\exists$ 'good' 1:1 correspondence characterized by endoscopies.

## Endoscopy (example)

Let

$$s = s_1 = \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix} \in C_\phi = \mathrm{Cent}(\phi, PGL(2, \mathbb{C})).$$

Then

$$^L H_s = \mathrm{Cent}(s, PGL(2, \mathbb{C}))^0 \cdot \phi(W_F)$$

is the $L$-group of $H_s = \ker[K_1^\times \xrightarrow{Norm_{K_1/F}} F^\times]$.

$^0$ means a connected component of 1.

$K_1$ is a subfield of $K$ corresponding to $\langle \tau_1 \rangle$.

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$L$-parameter $\phi$ factors through $^L H_s$,

$$\phi : W_F \stackrel{\phi_{H_s}}{\longrightarrow} {}^L H_s \longrightarrow {}^L G.$$

Therefore $L$-parameter $\phi_{H_s} : W_F \longrightarrow {}^L H_s$ defines a character $\pi(\phi_{H_s})$ of $H_s$.

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Therefore $L$-parameter $\phi_{H_s} : W_F \longrightarrow {}^L H_s$ defines a character $\pi(\phi_{H_s})$ of $H_s$.

$\exists$ arrangement $\pi_1, \cdots, \pi_4$ such that

$$\begin{aligned}
1 \cdot J(\pi_1) \; +1 \cdot J(\pi_2) \; +1 \cdot J(\pi_3) \; +1 \cdot J(\pi_4) &= c_0 \cdot \mathrm{Tran}_G^G J(\phi), \\
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\end{aligned}$$

$$(s_3 = s_1 s_2).$$

$J(\ )$: distribution character.

$J(\phi_{H_s}) = J(\pi(\phi_{H_s}))$.

$\mathrm{Tran}_{H_s}^G$: endoscopic transfer (defined modulo constant).
stable distribution on $H_s \longrightarrow$ invariant distribution on $G$

$c_0, c_1, c_2, c_3 \in \mathbb{C}^\times$: non-zero constant.

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$(s_3 = s_1 s_2)$.

Case $s = 1$

$H_s = G$.

We put $J(\phi) = J(\pi_1) + J(\pi_2) + J(\pi_3) + J(\pi_4)$.

$\mathrm{Tran}_G^G$: the identity map. ($c_0 = 1$.)

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**Character relation**

$$\sum_{i=1}^4 \rho_{\pi_i}(s) J(\pi_i) = c \cdot \mathrm{Tran}_{H_s}^G J(\phi_{H_s}).$$

$\exists$ 'good' correspondence.

$$\begin{array}{ccc} \Pi_\phi(G) & \longleftrightarrow & \Pi(C_\phi) \\ \pi_i & \longleftrightarrow & \rho_{\pi_i} \end{array}$$

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## What about inner forms ?

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We have

$$\text{Res}_{D^1}^{D^\times} \pi(\tilde{\phi}) = \pi \oplus \pi = 2\pi.$$

$$C_\phi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Character relation.

$$
\begin{aligned}
2 \cdot J(\pi) &= c_0 \cdot \mathrm{Tran}^G_{SL(N)} J(\phi). \\
0 \cdot J(\pi) &= c_1 \cdot \mathrm{Tran}^G_{H_{s_1}} J(\phi_{H_{s_1}}). \\
0 \cdot J(\pi) &= c_2 \cdot \mathrm{Tran}^G_{H_{s_2}} J(\phi_{H_{s_2}}). \\
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$\rho_\pi$: virtual character of $C_\phi$ !?

( $J(\phi) = J(\pi_1) + J(\pi_2) + J(\pi_3) + J(\pi_4)$. )

## Modification due to Vogan (and Kottwitz)

D. Vogan Jr., "The local Langlands conjecture." Representation theory of groups and algebras, 305–379, Contemp. Math., 145, (1993).

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Let $S_\phi$ be the pull-back of $C_\phi$ in $SL(2,\mathbb{C})$.

$$SL(2,\mathbb{C}) \longrightarrow PGL(2,\mathbb{C}) \supset C_\phi.$$

$Z_\phi = \{I_2, -I_2\} \subset S_\phi$.

## Modification due to Vogan (and Kottwitz)

D. Vogan Jr., "The local Langlands conjecture." Representation theory of groups and algebras, 305–379, Contemp. Math., 145, (1993).

Let $S_\phi$ be the pull-back of $C_\phi$ in $SL(2,\mathbb{C})$.

$$SL(2,\mathbb{C}) \longrightarrow PGL(2,\mathbb{C}) \supset C_\phi.$$

$Z_\phi = \{I_2, -I_2\} \subset S_\phi$.

Then $S_\phi$ is the quaternion group.

$\exists!$ 2 dimensional irreducible representation of $S_\phi$ with the central character $sgn$ on $Z_\phi$.

Character relation.

$$
\begin{aligned}
2 \cdot J(\pi) &= c_0 \cdot \mathrm{Tran}^G_{H_{I_2}} J(\phi). \\
-2 \cdot J(\pi) &= c_0' \cdot \mathrm{Tran}^G_{H_{-I_2}} J(\phi). \\
0 \cdot J(\pi) &= c_1 \cdot \mathrm{Tran}^G_{H_{s_1}} J(\phi_{H_{s_1}}). \\
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2 dimensional irreducible representation of the quaternion group $S_\phi$ with the central character $sgn$ on $Z_\phi$ !

## Character relation

$$\text{trace}\, \rho_\pi(s)\, J(\pi) = c \cdot \text{Tran}_{H_s}^{G}\, J(\phi_{H_s}).$$

$$\Pi_\phi(D^1) = \{\pi\} \longleftrightarrow \Pi(S_\phi, sgn) = \{\rho_\pi\}.$$

Remark that

$\text{Res}_{D^1}^{D^\times}\, \pi(\tilde{\phi}) = \pi \oplus \pi = 2\pi$, and $\rho_\pi$ is 2 dimensional.

Hasse invariant of $D$ is $\frac{1}{2}$, and the central character of $\rho_\pi$ on $Z_\phi$ is $sgn$.

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# §2 Main Theorem

$G$: inner form of $SL(N)$.

($G = \ker[GL_m(D) \overset{Reduced\,Norm}{\longrightarrow} F^{\times}]$, $D$: division algebra.)

$\tilde{G} = GL_m(D)$.

**Kottwitz map (Hasse invariant)**

$$\{\text{inner forms}\} \simeq H^1(F, PGL(N)) \simeq \text{char. of } Z_{SL(N,\mathbb{C})}$$

$$G \longrightarrow \chi_G$$

$\mathcal{S}_\phi = S_\phi / S_\phi^0$.

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**Theorem** **(H.–Saito).** *Let $\phi$ be an $L$-parameter or an $A$-parameter then there exist a 1:1 correspondence*

$$\Pi_{\phi}(G) \longleftrightarrow \Pi(\mathcal{S}_{\phi}, \chi_G)$$

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This is a generalization of [Labesse–Langlands].

This gives examples of Vogan's modified conjecture.

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If $G = SL(N)$, then this theorem is an easy consequence of [Henniart–Herb], because $\mathrm{Res}_{G}^{\tilde{G}}\, \pi(\tilde{\phi})$ is multiplicity free. So this result is not new in this case.

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If we are satisfied with the weaker statement
"$\rho_\pi$ is a virtual character of $C_\phi$",
then the proof is not difficult. (Most part has been proved by Henniart–Herb and Waldspurger.)

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$a_s$: 1-cocycle of $W_F$ in $Z_{\tilde{\hat{G}}} \simeq \mathbb{C}^\times$.

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Then $a_s$ defines a one dimensional character $\omega_s$ of $\tilde{G}$, and we have

$$\pi(\tilde{\phi}) \otimes \omega_s \simeq \pi(\tilde{\phi}).$$

Put

$$X(\pi(\tilde{\phi})) = \{\omega : \text{ character of } \tilde{G} \mid \pi(\tilde{\phi}) \otimes \omega \simeq \pi(\tilde{\phi})\}.$$

Then

$$\begin{array}{ccccccccc} 1 \longrightarrow \mathcal{Z}_\phi \longrightarrow & \mathcal{S}_\phi & \longrightarrow & X(\pi(\tilde{\phi})) & \longrightarrow 1 & \quad \text{(exact)} \\ & s & \longrightarrow & \omega_s & & \end{array}$$

For $\omega \in X(\pi(\tilde{\phi}))$, let $I_\omega$ be an intertwining operator on $V_{\pi(\tilde{\phi})}$ such that,

$$\pi(\tilde{\phi}) \circ I_\omega = I_\omega \circ (\pi(\tilde{\phi}) \otimes \omega).$$

($I_\omega$ is defined up to a scalar factor.)

Define $e(I_{\omega_1}, I_{\omega_2}) \in \mathbb{C}^\times$ by

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**Theorem** **(H.–Saito).** *For* $s_1, s_2 \in \mathcal{S}_\phi$,

$$e(I_{\omega_{s_1}}, I_{\omega_{s_1}}) = \chi_G(s_1\, s_2\, s_1^{-1}\, s_2^{-1}).$$

**Corollary.** $\exists$ *homomorphism* $\Lambda : \mathcal{S}_\phi \longrightarrow \mathrm{Aut}(V_{\pi(\tilde{\phi})})$ *such that*

$$\begin{aligned} \Lambda(s) &\in \mathbb{C}^\times \cdot I_\omega, & s &\in \mathcal{S}_\phi, \\ \Lambda(z) &= \chi_G(z) \cdot \mathrm{Id}, & z &\in \mathcal{Z}_\phi. \end{aligned}$$

Remark that $\chi_G$ corresponds to the class field theory.

The action of $\mathcal{S}_\phi$ on $V_{\pi(\tilde{\phi})}$ is defined by $\Lambda$.

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We prove the equation $e(I_{\omega_{s_1}}, I_{\omega_{s_1}}) = \chi_G(s_1\, s_2\, s_1^{-1}\, s_2^{-1})$ from

**Theorem (Labesse–Langlands, Henniart–Herb, Badulescu, H.–Saito).** *Let $\phi$ be an $A$-parameter. Then*

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$J^{\omega_s}(\pi(\tilde{\phi})) = \mathrm{trace}(\pi(\tilde{\phi}) \circ I_{\omega_s})$: twisted character.

$\tilde{H}_s$: (twisted) endoscopic group for $\tilde{G} = GL_m(D)$ given by $s$.

$\tilde{\phi}_{\tilde{H}_s}$: $A$-parameter for $\tilde{H}_s \longleftarrow \tilde{\phi}$.

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$N = 2$: by Labesse–Langlands (1979).

$\tilde{G} = GL(N)$ and $\phi$ is tempered: by Henniart–Herb (1995).

$s = 1$: by Badulescu (preprint)
nontempered: Aubert–Zelevinsky involution.

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We prove this theorem by using the simple trace formula We need to use the fundamental lemma and the transfer theorem due to Waldspurger.

We also use the matching results (supercuspidal representations) due to Henniart–Herb.

We imitate the arguments of Badulescu.

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In order to show the equation, we have to study the automorphism of the endoscopy.

Especially, we have to study the effect of the automorphism on the transfer factor.

Parts of the arguments are also contained in

J. Arthur, "A note on $L$-packets."

Pure Appl. Math. Q. 2 (2006), no. 1, 199–217.