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# Graded Hecke algebras

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Based on joint work with D. Barbasch, P. Trapa.

# 1 Unequal parameters graded Hecke algebra

| **Type** | **c** | **geometric** |
|---|---|---|
| $B_n$ | $c(\epsilon_1 - \epsilon_2) = 1$<br>$c(\epsilon_n) = c > 0$ | $c \in \mathbb{Z}$ $(SO(m, \mathbb{C}))$<br>$c \in 1/2 + \mathbb{Z}$ $(Sp(m, \mathbb{C}))$<br><br>$c \in \pm 1/4 + \mathbb{Z}$ $(Spin(m, \mathbb{C}))$<br><br>$c \notin 1/2\mathbb{Z}$ (Kato "exotic nilpotent cone" of $Sp(2n, \mathbb{C})$) |
| $G_2$ | $c(\alpha_s) = 1, c(\alpha_l) = c$ | $c = 3$ $(E_6)$ |
| $F_4$ | $c(\alpha_s) = 1, c(\alpha_l) = c$ | $c = 2$ $(E_7)$ |

There is also an independent "harmonic analysis" classification due to Opdam for the affine Hecke $\mathcal{H}$ in all (?) cases.

## 2 Classification of $G_{0,\chi}\backslash\mathfrak{g}_{1,\chi}$

Kawanaka 1987 classified the $G_{0,\chi}$ orbits on $\mathfrak{g}_{1,\chi}$.

There are finitely many orbits, so there is a unique open orbit.

The simplest example is when $\chi = \check{\rho}$. Then

$$\mathfrak{g}_{1,\check{\rho}} = \bigoplus_{\alpha\in\Pi} \mathbb{C}\cdot X_\alpha,$$

where $X_\alpha$ are root vectors, and $G_{0,\check{\rho}}$ is the Cartan $H$.

So

$$\begin{aligned} &G_{0,\check{\rho}}\backslash\mathfrak{g}_{1,\chi} \longleftrightarrow 2^{\Pi}, \\ &\mathcal{O}_P = \sum_{\alpha\in\Pi_P} \mathbb{C}^*\cdot X_\alpha \longleftarrow \Pi_P \subset \Pi. \end{aligned}$$

In this case, all orbits have smooth closure. Moreover, the closure ordering is just the inclusion of subsets of simple roots.

In general, Lusztig 2006 gives a reformulation of Kawanaka's classification by "good" parabolic subalgebras, and an algorithm for computing KL polynomials.

## 3 Zelevinski's matching

$G = GL(n, \mathbb{C})$, $W = S_n$. The geometric parameters for $\mathbb{H}$ can be viewed as *multisegments*. The geometric parameters for category $\mathcal{O}$ correspond to elements of $S_n$. There matching of geometric parameters so that the KL polynomials are the same.

$$G_{0,\chi} \backslash \mathfrak{g}_{1,\chi} \hookrightarrow S_n$$

assigns to every multisegment a product of cycle permutations.

For example, at $\chi = \check{\rho}$ for $GL(3)$:

| Multisegments | $S_n$ |
|---|---|
| ● ● ● | () |
| ●–● ● | (12) |
| ● ●–● | (23) |
| ●–●–● | (123) |

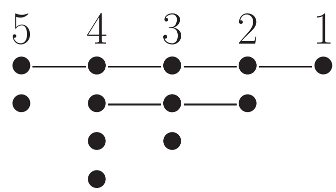
## 4 Matching for $GL(n, \mathbb{R})$ (with P. Trapa)

### Example of parametrization

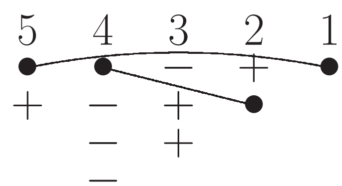
Take $\chi = (5, 5, 4, 4, 4, 4, 3, 3, 3, 2, 2, 1)$. So

$$G_{0,\chi} \simeq GL(2) \times GL(4) \times GL(3) \times GL(2) \times GL(1)$$

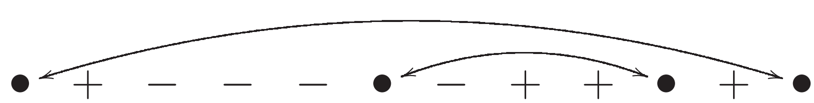
For instance, one multisegment is




### Defining $\Phi \,:\, G_{0,\chi} \backslash \mathfrak{g}_{1,\chi} \longrightarrow \Sigma_{\pm}(n)$




Flatten:



## 5 Reducibility of standard modules

We consider the Hecke algebra with equal parameters. Let $\Pi_P \subset \Pi$ be maximal, $\{\alpha\} = \Pi - \Pi_P$. Let $\check{\omega}$ denote the fundamental coweight for $\alpha$.

If $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is the corresponding Lie algebra, $\sigma$ is attached to a map $\mathfrak{s}l(2) \hookrightarrow \mathfrak{m}$.

Then $\mathfrak{n}$ is an $\mathfrak{s}l(2)$-module, via the adjoint action of $\mathfrak{m}$.

The coweight $\check{\omega}$ commutes with the $\mathfrak{s}l(2)$. Decompose $\mathfrak{n}$ as $\mathfrak{n} = \oplus_{i=1}^{k} \mathfrak{n}_i$, where $\mathfrak{n}_i$ is the $i$-eigenspace of $\check{\omega}$.

Decompose each $\mathfrak{n}_i$ into simple $\mathfrak{s}l(2)$-modules $\mathfrak{n}_i = \oplus_j (d_{ij})$, where $(d)$ is the simple $\mathfrak{s}l(2)$-module of dimension $d$.

The following result is known from Muić-Shahidi; the Hecke algebra proof follows from the geometric classification.

**Theorem 5.1** (M-S, Barbasch-C)**.** *Let $\sigma$ be a generic tempered module. Then the reducibility points $\nu > 0$ of $X(P, \sigma, \nu)$ are*

$$\left\{ \frac{d_{ij} + 1}{2i} \right\}_{i,j}.$$

*Proof.* Let $\chi$ be the central character.

Let $\mathcal{O}^{\chi}(P, \sigma)$ be the orbit corresponding to $\overline{X}(P, \sigma, \nu)$.

The generic module is parametrized by the trivial local system on the *open* orbit $\mathcal{O}^{\chi}_{op}$. So $X(P, \sigma, \nu)$ is irreducible if and only if

$$\mathcal{O}^{\chi}(P, \sigma) = \mathcal{O}_{op}.$$

From Lusztig's algorithm 2006:

$$\dim \mathcal{O}^{\chi}(P, \sigma) = \dim \mathfrak{g}_{0,\chi} - \dim(\mathfrak{g}_{0,\chi} \cap \bar{\mathfrak{p}}) + \dim(\mathfrak{g}_{1,\chi} \cap \bar{\mathfrak{p}}).$$

But $\dim \mathcal{O}_{op} = \dim \mathfrak{g}_{1,\chi}$.

Write the roots $\Delta(\bar{p}) = \Delta(\mathfrak{m}) \cup \Delta(\bar{\mathfrak{n}})$. The dimension argument says that $\mathcal{O}^{\chi}(P, \sigma) = \mathcal{O}_{op}$ if and only if

$$\#\{\alpha \in \Delta(\mathfrak{n}) : \langle \alpha, \chi \rangle = 1\} = \#\{\alpha \in \Delta(\mathfrak{n}) : \langle \alpha, \chi \rangle = 0\}.$$

Consider the rational function of $\nu$:

$$\prod_{\alpha \in \Delta(\mathfrak{n})} \frac{1 - \langle \alpha, \chi \rangle}{\langle \alpha, \chi \rangle}.$$

The reducibility points are given by the zeros of this function.

$\square$

One partial result for non-generic data:

**Corollary 5.2.** *Let $\sigma'$ be a tempered module, in the same L-packet with the generic tempered $\sigma$. Then the reducibility points of $X(P, \sigma', \nu)$ are a subset of the reducibility points of $X(P, \sigma, \nu)$.*

## 6 Generic unitary dual

The spherical principal series $X(\chi)$ is reducible if and only if $\langle \alpha, \chi \rangle = 1$, for some positive root $\alpha$.

**Theorem 6.1** (Barbasch-C.)**.** *The set of unitary spherical generic parameters $CS(0)$ is a union of $k$ simplices (alcoves) in the dominant Weyl chamber, where:*

$$\begin{aligned} G_2&: \; k = 2 \\ F_4&: \; k = 2 \\ E_6&: \; k = 2 \\ E_7&: \; k = 8 \\ E_8&: \; k = 16. \end{aligned}$$

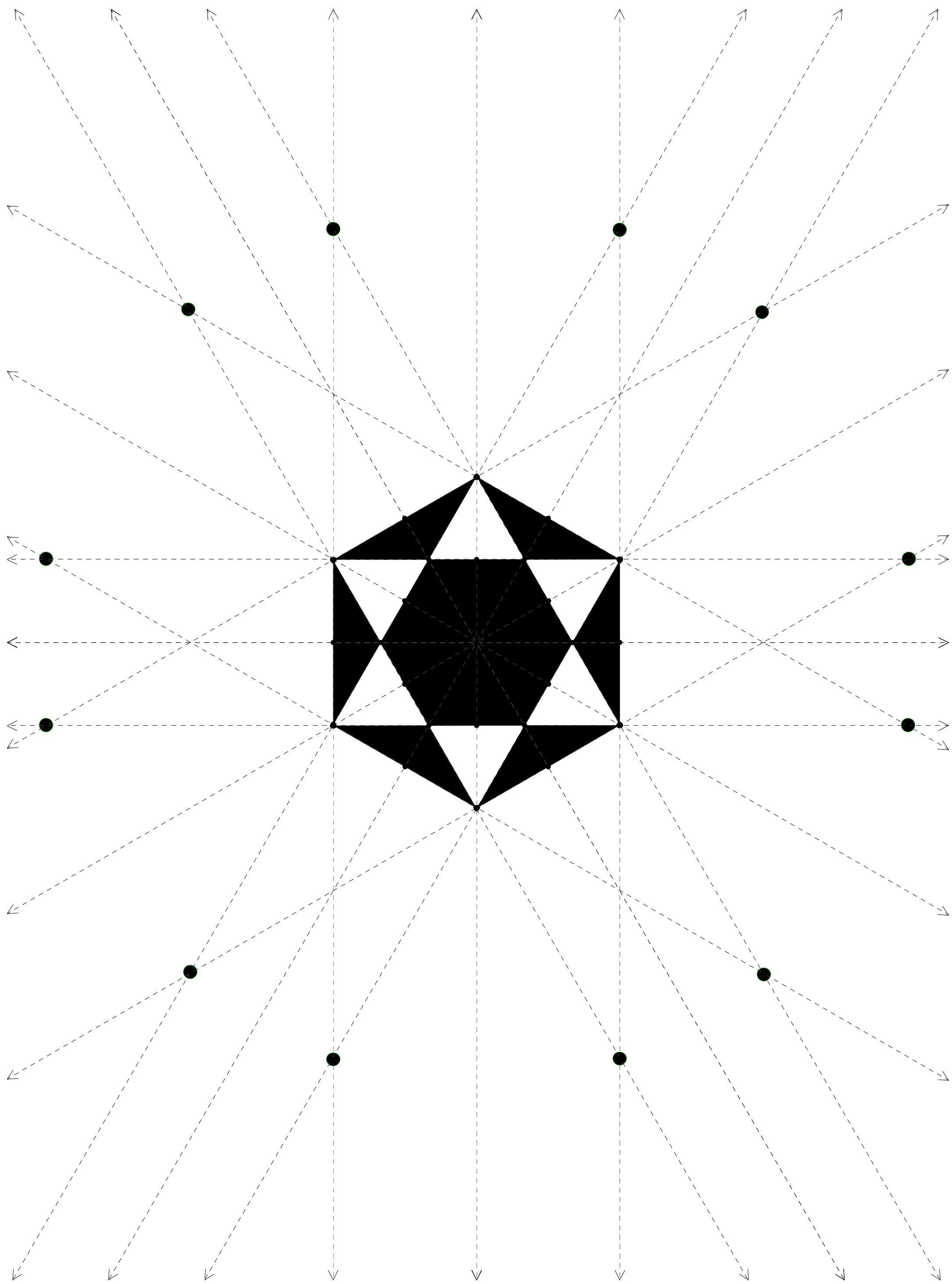


Figure 1: Generic unitary dual of $G_2$

One partitions the spherical unitary dual into pieces $CS(\mathcal{O})$ parametrized by nilpotent orbits.

Let $Exc$ denote the following set of nilpotent orbits:

$$Exc = \{\underbrace{A_1\widetilde{A}_1}_{F_4}, \underbrace{A_2 3A_1}_{E_7}, \underbrace{A_4A_2A_1, A_4A_2, D_4(a_1)A_2, A_3 2A_1, A_2 3A_1, 4A_1}_{E_8}\}.$$

(The notation is as in the Bala-Carter classification.)

Then the generic unitary dual of $\mathbb{H}$ can be described as follows.

**Theorem 6.2** (Barbasch-C.)**.**

1. *If* $\mathcal{O} \notin Exc$, *then*

$$CS(\mathcal{O}) = CS_{\mathfrak{z}(\mathcal{O})}(0).$$

2. *If* $\mathcal{O} \in Exc$, *and* $\mathcal{O} \neq (4A_1 \subset E_8)$, *then* $CS(\mathcal{O}) \subsetneq CS_{\mathfrak{z}(\mathcal{O})}(0)$.
3. *If* $\mathcal{O} = (4A_1 \subset E_8)$, *then* $CS(\mathcal{O}) \supsetneq CS_{\mathfrak{z}(\mathcal{O})}(0)$.

Note that by the Iwahori-Matsumoto involution, this gives the spherical unitary dual as well.