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Highest weight categories and representations of W-algebras.

affine W-algebras = VAs (vertex algebra) ⊃ Virasoro VA

W after V = Virasoro alg, W

- non-linearlity
- Many W-algebras can be constructed by BRST reduction

Feigin-Frenkel, de-Boer Tjin, ... Kac-Roan-Wakimoto '03

$\mathfrak{g}$: simple

$x \in \mathfrak{g}^*$ nilpotent $\rightsquigarrow W^k(\mathfrak{g}, x) = \underbrace{H^0_x}_{\text{certain BRST cohomology}}(V^k(\mathfrak{g}))$, $V^k(\mathfrak{g})$: $\mathfrak{g}_{aff}$ at level k

$k \in \mathbb{C}$

Example (1) $W^k(\mathfrak{g}, 0) = V^k(\mathfrak{g})$ ← VA ass with $\mathfrak{g}_{aff}$ at level k.
BRST coh. vanishes.

$\{V^k(\mathfrak{g})\text{-modules}\} = \{\text{smooth rep. } \mathfrak{g}_{aff} \text{ at level } k\}$

(2) $W^k(sl_2, x) = \begin{cases} \text{Virasoro VA} & k \neq -2 \\ Z(V^{-2}(sl_2)) & k = -2 \end{cases}$ Feigin-Frenkel, $x \neq 0$

✡ $W^{-h^\vee}(\mathfrak{g}, x_{prin}) = Z(V^{-h^\vee}(\mathfrak{g}))$ $h^\vee$ = dual Coxeter #

(3) $W^k(sl_3, x_{prin})$ $k \neq -3$.

generators : L_n, W_n $(n \in \mathbb{Z})$

relations: L_n generates Virasoro

$[L_m, W_n] = (m-n) W_{m+n}$

$[W_m, W_n] = (m-n)\{\frac{1}{15}(m+n+2)(m+n+3) - \frac{1}{6}(m+2)(n+2)\} L_{m+n}$
$+ \frac{16}{22+5C(k)}(m-n)\Lambda_{m+n} + \frac{C(k)}{360} m(m^2-1)(m^2-4)\delta_{m+n,0}\, C$

$\Lambda_m = \sum_{k\in\mathbb{Z}} :L_{m-k}L_k: - \frac{3}{10}(n+2)(n+3)L_n$

V: VA $\rightsquigarrow$ Zh(V) : Zhu's alg. (unital asso. alg.)

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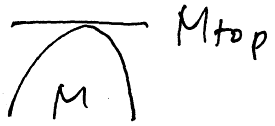
$\mathcal{U}(V) = \bigoplus_{d\in\mathbb{Z}} \mathcal{U}(V)_d$: universal env. alg. of V

$\Rightarrow Zh(V) = \mathcal{U}(V)_0 \Big/ \overline{\sum_{p>0} \mathcal{U}(V)_{-p}\,\mathcal{U}(V)_p}$

$\left\{\begin{matrix}\text{graded simple}\\ V\text{-modules}\end{matrix}\right\} \xleftrightarrow{1:1} \{\text{simple } Zh(V)\text{-modules}\}$



$\longmapsto \quad M_{top}$

ex. $Zh(V^k(\mathfrak{g})) \simeq U(\mathfrak{g})$

$Zh(\text{Virasoro}) \simeq \mathbb{C}[L_0] \simeq Z(U(\mathfrak{sl}_2))$

$Zh(W^k(\mathfrak{g}, x_{prin})) \simeq Z(\mathfrak{g})$

$Zh_\sigma(W^k(\mathfrak{g}, x)) \simeq W^{fin}(\mathfrak{g}, x)$: finite W-alg.

Ramond type, twisted one De Sole-Kac $\quad \| \quad$ $\mathrm{End}_{U(\mathfrak{g})}(\text{Gelfand Graev}_x)^{op}$.

$GG_x = U(\mathfrak{g}) \otimes_{U(n)} \mathbb{C}_x \qquad GG_x$

Rem (1) $k \notin \mathbb{Q}$ $\left\{\begin{matrix}\text{(Ramond type twisted)}\\ \text{repr of } W^k(\mathfrak{g}, x)\end{matrix}\right\} \simeq W^{fin}(\mathfrak{g}, x)$-mod.

(2) $W^{fin}(\mathfrak{g}, x) \simeq \{M \mid (y - x(y))^r v = 0 \quad r \gg 0 \quad y \in m, v \in M\}$

Skryabin $\quad Wh(M) \longleftarrow M$

$E \longmapsto GG_x \otimes_{W^{fin}(\mathfrak{g},x)} E$

Assume that $\exists$ a good even grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ for x

$\exists e \in \mathfrak{g}_1$ s.t. ① $x = (e \mid ?)$ ② e is a Richardson element for $\mathfrak{g}_{>0} = \bigoplus_{j>0} \mathfrak{g}_j$

$GG_x = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{<0})} \mathbb{C}_x$

$W^{fin}(\mathfrak{g}, x) = \mathrm{End}_{U(\mathfrak{g})}(GG_x)^{op}$.

Likewise $\left\{\begin{matrix}\text{Raymond type twisted}\\ \text{repr. of } W^k(\mathfrak{g}, x)\end{matrix}\right\} = \left(\begin{matrix}\text{(untwisted) Repr of } W^k(\mathfrak{g}, x)\\ \text{with a slight change of}\\ \text{Hamiltonian}\end{matrix}\right)$

finite dimensional setting

$\mathcal{O}$: parabolic subcategory of BGG category of $\mathfrak{g}$ wrt $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j$

$H_0(\mathfrak{g}_{<0}, M \otimes \mathbb{C}_\chi) = (Wh(\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})))^*$

: $W^{fin}(\mathfrak{g}, \chi)$-module

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & W^{fin}(\mathfrak{g}, \chi)\text{-mod}. \\ \Psi & & \\ M & \longmapsto & \overline{H}_0^\chi(M) = H_0(\mathfrak{g}_{<0}, M \otimes \mathbb{C}_\chi) \end{array}$$

① (Kostant-Lynch) $\overline{H}^\chi_{i \neq 0}(M) = 0$ so $\overline{H}^\chi_0(?)$ is exact

② (Matumoto) $\overline{H}^\chi_0(L(\lambda)) \neq 0 \Leftrightarrow \mathrm{Dim}\, L(\lambda) = \dim \mathfrak{g}_{>0}$

③ If $\begin{cases} \chi \text{ is principal} & W^{fin}(\mathfrak{g}, \chi_{prin}) \simeq Z(\mathfrak{g}) \quad \text{Kostant} \\ \text{or} & \\ \mathfrak{g} \text{ type A} & \text{Brundan-Kleshev}. \end{cases}$

$\overline{H}^\chi_0(L(\lambda))$ is simple (or zero) and any f.d. simple $W^{fin}(\mathfrak{g}, \chi)$ appears in this way.

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affine setting

$\mathfrak{g}_{aff.} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$

$\cup$

$L\mathfrak{g}_{<0} = \mathfrak{g}_{<0} \otimes \mathbb{C}[t, t^{-1}]$ $\qquad \chi_{aff}: L\mathfrak{g}_{<0} \longrightarrow \mathbb{C}$

$\qquad\qquad\qquad\qquad y \otimes t^n \longmapsto \chi(y)\delta_{n,0}$

$\mathcal{O}^k_{aff} = \{M \in BGG^k(\mathfrak{g}_{aff}) \mid$

M is a direct sum of objects in $\mathcal{O}\}$

$$\begin{array}{ccc} \mathcal{O}^k_{aff} & \longrightarrow & W^k(\mathfrak{g}, \chi)\text{-mod} \\ \Psi & & \\ M & \longrightarrow & H^\chi_0(M) = H_{\frac{\infty}{2}+0}(L\mathfrak{g}_{<0}, M \otimes \mathbb{C}_{\chi_{aff}}) \end{array}$$

Remark $\mathcal{O}^k_{aff} \ni M = \bigoplus_{d \in \mathbb{C}} M_d \leftarrow$ d-eigenspace of $D \in \mathfrak{g}_{aff}$

$H^\chi_0(M) = \bigoplus_{d \in \mathbb{C}} H^0_\chi(M)_d$.

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$H_i^x(M)_{top}$: module of $Zhu_\sigma(W^k(\mathfrak{g},x)) = W^{fin}(\mathfrak{g},x)$

$\parallel$

$\begin{cases} \bar{H}_i^x(M_{top}) & i \geq 0 \\ 0 & i < 0 \end{cases}$

<u>Theorem</u> k any

(1) $H^x_{i\neq 0}(M) = 0 \quad \forall M \in \mathcal{O}^k_{aff}$

so $\mathcal{O}^k_{aff} \longrightarrow W^k(\mathfrak{g},x)\text{-mod}$

$M \longmapsto H^x_0(M)$

is exact

(2) $H^x_0(L(\lambda)) \neq 0$

$\Leftrightarrow H^x_0(L(\lambda))_{top} \neq 0$

$\Leftrightarrow \bar{H}^x_0(\underbrace{L(\lambda)_{top}}_{= \bar{L}(\bar{\lambda})}) \neq 0$

$\Leftrightarrow \mathrm{Dim}\, L(\lambda)_{top} = \dim \mathfrak{g} > 0$

(3) $H^x_0(L(\lambda))$ is <u>almost irred.</u> (or zero)

$\uparrow$

① $H^x_0(L(\lambda))$ is generated by $H^x_0(L(\lambda))_{top}$.

② There is no nonzero submodule of $H^x_0(L(\lambda))$ intersecting $H^x_0(L(\lambda))_{top}$ trivially

($H^x_0(L(\lambda))$ is irred. $\Leftrightarrow H^x_0(L(\lambda))_{top}$ is irred $/ W^{fin}(\mathfrak{g},x)$)

<u>Remark</u> ① By thm, if $\begin{cases} x: \text{principal} \\ \mathfrak{g} \text{ type } A \end{cases}$, then all h.w. repr. of $W^k(\mathfrak{g},x)$ (s.t. top part is f.d.) $\simeq H^0_x(\exists L(\lambda))$

② $ch(H^0_x(L(\lambda))) = \sum (-1)^i ch\, H^i_x(L(\lambda))$

$= \sum (-1)^i C^i_x(L(\lambda)) = chL(\lambda) \times \prod_{\substack{\alpha \in \hat{\Delta}^{re}_{aff,+} \\ \bar{\alpha} \in \pm\Delta_{>0}}} (1 - e^{-\alpha})$

known in terms of KL-poly. if $k \neq -h^\vee$ (Kashiwara-Tanisaki)

proves the conj of { Frenkel-Kac-Wakimoto, Kac-Roan-Wakimoto

③ $H^0_x(\text{integrable module}) = 0$

④ If $k = -h^\vee$ one can do the opposite argument (In this case $W^{-h^\vee}(\mathfrak{g}, x_{prin})$ is commutative

$\Rightarrow$ KK-ch formula Explicit ch. formula of $L(\lambda)$ at the level