On a generalization of Jacquet modules of degenerate principal series representations

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Jacquet modules

- $G = KA_0N_0$: a semisimple Lie group and its Iwasawa decomposition.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$: the complexification of the Lie algebra of $G = KA_0N_0$.
- V: a Harish-Chandra module (Fréchet representation).
- V': the continuous dual of V.
- $(V_{K\text{-finite}})^* = \mathrm{Hom}_{\mathbb{C}}(V_{K\text{-finite}}, \mathbb{C})$.

We define the following two modules:

$$J'(V) = \{v \in V' \mid \exists k,\ \mathfrak{n}_0^k v = 0\},$$
$$J^*(V) = \{v \in (V_{K\text{-finite}})^* \mid \exists k,\ \mathfrak{n}_0^k v = 0\}.$$

Notice that $J'(V) = J^*(V)$ (Casselman-Wallach). This is called the Jacquet module.

Generalized Jacquet modules

- $G = KA_0N_0$: a semisimple Lie group and its Iwasawa decomposition.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$: the complexification of the Lie algebra of $G = KA_0N_0$.
- V: a Harish-Chandra module (Fréchet representation).
- V': the continuous dual of V.
- $(V_{K\text{-finite}})^* = \mathrm{Hom}_{\mathbb{C}}(V_{K\text{-finite}}, \mathbb{C})$.
- η: a character of N_0

We define the following two modules:

$$J'_\eta(V) = \{v \in V' \mid \exists k,\ (\mathrm{Ker}\,\eta)^k v = 0\},$$
$$J^*_\eta(V) = \{v \in (V_{K\text{-finite}})^* \mid \exists k,\ (\mathrm{Ker}\,\eta)^k v = 0\}.$$

where $\eta\colon U(\mathfrak{n}_0) \to \mathbb{C}$ ($\mathbb{C}$-algebra homomorphism).

Motivations

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g},K}(V_{K\text{-finite}}, \mathrm{Ind}_{N_0}^{G}(1_{N_0})) &= \mathrm{Hom}_{\mathfrak{n}_0}(V_{K\text{-finite}}, 1_{N_0}) \\ &= \{v \in (V_{K\text{-finite}})^* \mid \mathfrak{n}_0 v = 0\} \\ &= H^0(\mathfrak{n}_0, (V_{K\text{-finite}})^*).\end{aligned}$$

"to know $H^0(\mathfrak{n}_0, (V_{K\text{-finite}})^*)$"

$\Longleftrightarrow$ "to know $\mathrm{Hom}_{\mathfrak{g},K}(V_{K\text{-finite}}, \mathrm{Ind}_{N_0}^{G}(1_{N_0}))$"

$\Longleftrightarrow$ "to know homomorphisms to the principal series representations"

However $V \mapsto H^0(\mathfrak{n}_0, (V_{K\text{-finite}})^*)$ is NOT exact.

$\Rightarrow V \mapsto J^*(V)$: exact.

$H^0(\mathfrak{n}_0, (V_{K\text{-finite}})^*) = H^0(\mathfrak{n}_0, J^*(V))$.

Whittaker models

"Generalization"

- $J^* \rightsquigarrow J^*_\eta$
- $\{v \in (V_{K\text{-finite}})^* \mid \mathfrak{n}_0 v = 0\} \rightsquigarrow \{v \in (V_{K\text{-finite}})^* \mid (\operatorname{Ker}\eta)v = 0\}$
- $\operatorname{Hom}_{\mathfrak{g},K}(V_{K\text{-finite}}, \operatorname{Ind}_{N_0}^G(1_{N_0})) \rightsquigarrow \operatorname{Hom}_{\mathfrak{g},K}(V_{K\text{-finite}}, \operatorname{Ind}_{N_0}^G(\eta))$.
- "homomorphisms to the principal series representations"
  $\rightsquigarrow$ "homomorphisms to $\operatorname{Ind}_{N_0}^G \eta$" : Whittaker model.
- J'_η corresponds to the moderate growth homomorphism.

degenerate principal series representations

- $P = MAN$: a parabolic subgroup of G and its Langlands decomposition.
- $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$: the complexification of the Lie algebra of $P = MAN$.
- σ: a finite-length representation of M (Fréchet representation).
- $\lambda \in \mathfrak{a}^* = (\mathrm{Lie}(A)_{\mathbb{C}})^*$: a one-dimensional representation of A.
- $\rho(H) = (\mathrm{Tr}\,\mathrm{ad}(H)|_{\mathfrak{n}})/2$ for $H \in \mathfrak{a}$: one dimensional representation of A.

Put

$$I(\sigma, \lambda) = C^\infty\text{-}\,\mathrm{Ind}_P^G(\sigma \otimes (\lambda + \rho))$$

In this talk we consider $J'_\eta(I(\sigma, \lambda))$ and $J^*_\eta(I(\sigma, \lambda))$.

Example: $G = SL(2,\mathbb{R})$

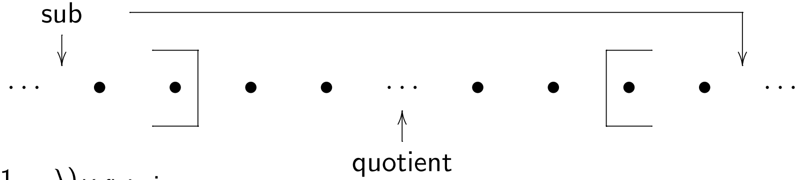
$G = SL(2,\mathbb{R})$, $K = SO(2)$, $P = P_0$: minimal parabolic
$\eta = 0$: trivial representation.

- λ: not integral or $\lambda/2$ is integral.
  $\Rightarrow I(1,\lambda)$ and $I(1,-\lambda)$ are irreducible and $I(1,\lambda) \simeq I(1,-\lambda)$.
  $\Rightarrow \dim \operatorname{Hom}(I(1,\lambda), I(1,\lambda)) = 1$
  $\dim \operatorname{Hom}(I(1,\lambda), I(1,-\lambda)) = 1$
  $\Rightarrow \dim H^0(\mathfrak{n}_0, (I(1,\lambda)_{K\text{-finite}})^*) = 2.$

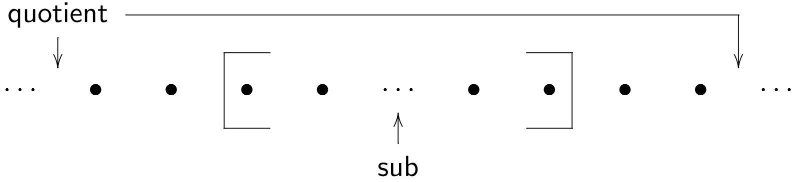
Example: $G = SL(2,\mathbb{R})$

- λ: integral, $\lambda/2$ is not integral, λ is dominant.

$I(1,\lambda)_{K\text{-finite}}$:



$I(1,-\lambda)_{K\text{-finite}}$:



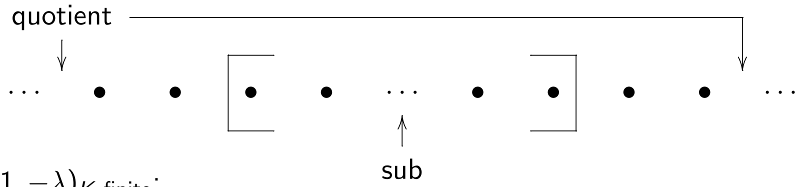
$\Rightarrow \dim \operatorname{Hom}(I(1,\lambda), I(1,\lambda)) = 1$, $\dim \operatorname{Hom}(I(1,\lambda), I(1,-\lambda)) = 1$

$\Rightarrow \dim H^0(\mathfrak{n}_0, (I(1,\lambda)_{K\text{-finite}})^*) = 2$.

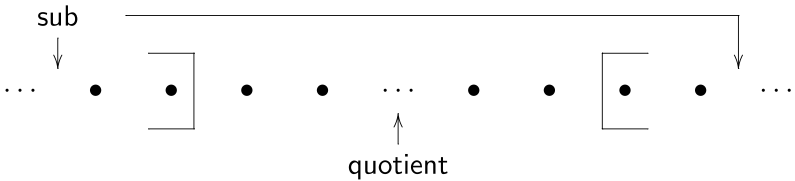
Example: $G = SL(2,\mathbb{R})$

- λ: integral, $\lambda/2$ is not integral, λ is anti-dominant.

$I(1,\lambda)_{K\text{-finite}}$:



$I(1,-\lambda)_{K\text{-finite}}$:



$\Rightarrow \dim \mathrm{Hom}(I(1,\lambda), I(1,\lambda)) = 1$, $\dim \mathrm{Hom}(I(1,\lambda), I(1,-\lambda)) = 2$

$\Rightarrow \dim H^0(\mathfrak{n}_0, (I(1,\lambda)_{K\text{-finite}})^*) = 3$.

Main Theorem

- W: the little Weyl group of G.
- $W(M) = \{w \in W \mid w(\text{positive roots of } M) \subset \text{positive roots of } G\}$.
- $W(M) = \{w_1, \dots, w_r\}$ such that $\bigcup_{i<j} N_0 w_i P/P \subset G/P$ is a closed subset.

Theorem

If η is not unitary then $J'_\eta(I(\sigma, \lambda)) = 0$.
Assume that η is unitary. There exists a filtration $\{I_i\}$ of $J'_\eta(I(\sigma, \lambda))$ such that

1. $I_i/I_{i-1} \neq 0$ *if and only if* $\eta|_{\mathrm{Ad}(w_i)\mathfrak{n}\cap\mathfrak{n}_0} = 0$ *and* $J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)) \neq 0$
2. *If* $I_i/I_{i-1} \neq 0$ *then* $I_i/I_{i-1} \simeq T_{w_i,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)))$.

where $T_{w_i,\eta}$ is the twisting functor (precise explanation will be appeared).

Main Theorem (2)

V: $U(\mathfrak{g})$-module.

- $C(V) = ((V^*)_{\mathfrak{h}\text{-finite}})^*$ ($\mathfrak{h}$: Cartan subalgebra)
  If $V = \bigoplus_{\nu \in \mathfrak{h}^*} V_\nu$ ($\mathfrak{h}$-weight decomposition) then $C(V) = \prod_{\nu \in \mathfrak{h}^*} V_\nu$.
- $\Gamma_\eta(V) = \{v \in V \mid \exists k,\ (\operatorname{Ker}\eta)^k v = 0\}$. ($J'_\eta(V) = \Gamma_\eta(V')$, $J^*_\eta(V) = \Gamma_\eta((V_{K\text{-finite}})^*)$.

Theorem

There exists a filtration $\{\widetilde{I_i}\}$ of $J^*_\eta(I(\sigma,\lambda))$ such that $\widetilde{I_i}/\widetilde{I_{i-1}} \simeq \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes (\lambda + \rho)))))$.

where $T_{w_i} = T_{w_i,0}$: twisting functor.

Bruhat filtration

An element of $J'_\eta(I(\sigma,\lambda))$ is regarded as a distribution on G/P with values in some vector bundle.
Put

$$I_i = \{x \in J'_\eta(I(\sigma,\lambda)) \mid \operatorname{supp} x \subset \bigcup_{j\le i} N_0 w_j P/P\}$$

Then $0 = I_0 \subset I_1 \subset \cdots \subset I_r = J'_\eta(I(\sigma,\lambda))$: Bruhat filtration.
$w_i\overline{N}w_i^{-1} \cap N_0 \simeq N_0 w_i P/P \subset w_i\overline{N}w_i^{-1} \simeq w_i\overline{N}P/P \subset G/P$: open subset
$N_0 w_j P/P \cap w_i\overline{N}P/P = \emptyset$ for $j < i$.

$$\begin{aligned}\Rightarrow I_i/I_{i-1} &\hookrightarrow \{x \in \mathcal{T}(w_i\overline{N}w_i^{-1}) \mid \operatorname{supp} x \subset w_i\overline{N}w_i^{-1} \cap N_0\} \\ &= U(\operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{T}(w_i\overline{N}w_i^{-1} \cap N_0)\end{aligned}$$

where $\mathcal{T}$ means tempered distribution with values in $(\sigma \otimes (\lambda+\rho))'$.

Bruhat filtration (2)

Lemma

$\mathrm{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0$ *acts* $U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{T}(w_i \overline{N} w_i^{-1} \cap N_0)$ *locally nilpotent.*

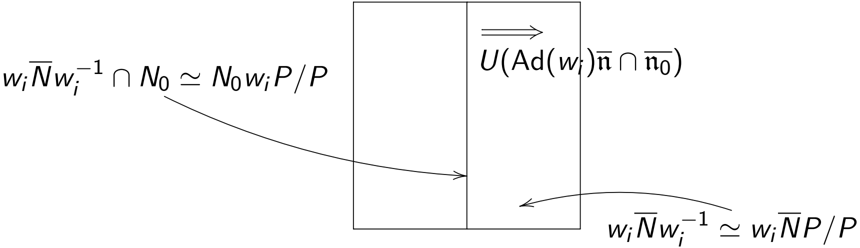
Hence $\eta|_{\mathrm{Ad}(w_i)\mathfrak{n}\cap\mathfrak{n}_0} \neq 0 \Rightarrow I_i/I_{i-1} = 0$ (a part of the theorem).
From now on, we assume that $\eta|_{\mathrm{Ad}(w_i)\mathfrak{n}\cap\mathfrak{n}_0} = 0$.

Bruhat filtration (3)

Lemma

$$\{x \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{T}(w_i\overline{N}w_i^{-1} \cap N_0) \mid \exists k,\ (\mathrm{Ker}\,\eta)^k x = 0\}$$
$$= U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{P}(w_i\overline{N}w_i^{-1} \cap N_0)\eta^{-1} \otimes J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)).$$

$\mathcal{P}$ is the polynomial ring.



Bruhat filtration (4)

We have $I_i/I_{i-1} \hookrightarrow$
$U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{P}(w_i\overline{N}w_i^{-1} \cap N_0)\eta^{-1} \otimes J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)) =: I'_i$.

Lemma

This map is surjective.

The proof is based on the meromorphic continuation.

meromorphic continuation

For $f \in \mathcal{P}(w_i\overline{N}w_i^{-1} \cap N_0)$, $u' \in J'_{w_i^{-1}}(\sigma \otimes (\lambda + \rho))$, we define the distribution $\delta(f, u')$ on $w_i\overline{N}P/P$ by

$$\langle \delta(f, u'), \varphi \rangle = \int_{w_i\overline{N}w_i^{-1} \cap N_0} f(n)u'(\varphi(nw_i))\eta(n)^{-1} dn$$

for $\varphi \colon w_i\overline{N}P/P \to \sigma \otimes (\lambda + \rho)$, $\operatorname{supp}\varphi$: compact.

$T\delta(f, u') \leftrightarrow T \otimes f\eta^{-1} \otimes u'$.

Lemma

1. *If λ is sufficiently large then the above integral absolutely converges for $\varphi \in I(\sigma, \lambda)$ and defines the distribution with holomorphic parameter.*
2. *$\delta(f, u')$ has a meromorphic continuation for $\lambda \in \mathfrak{a}^*$.*

This lemma is well-known.

proof of the surjectivity

Lemma

$x \in I'_i$, $\exists u_t \in I_i \subset J'_\eta(I(\sigma, \lambda + t\rho))$ with holomorphic parameter t defined near $t = 0$ such that $u_0 = x$ on $w_i\overline{N}P/P$.

Proof.

Induction on i. Fix $x \in I'_i$.
$\exists u'_t \in I_i$ with meromorphic parameter t such that $u'_0 = x$ on $w_i\overline{N}P/P$.
$u'_t = \sum_{s=-p}^{\infty} u^{(s)} t^s$: Laurent series.

1. $p = 0 \Rightarrow$O.K.
2. $p > 0 \Rightarrow u^{(-p)} \in I'_{i-1}$
   $\Rightarrow \exists u''_t \in I_{i-1}$ s.t. $u''_0 = u^{(-p)}$ on $w_{i-1}\overline{N}P/P$
   $\Rightarrow$ Replace u'_t to $u'_t - t^{-p}u''_t$ then $p \mapsto p - 1$.

□

(Generalized) twisting functors

$w \in W$, $\{e_1, \dots, e_l\}$: basis of $\mathrm{Ad}(w)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$, e_i: root vector w.r.t. $\mathfrak{h}$.

$$S_{w,\eta} = (U(\mathfrak{g})[(e_1 - \eta(e_1))^{-1}]/U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} \cdots \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})[(e_l - \eta(e_l))^{-1}]/U(\mathfrak{g}))$$

For X: $U(\mathfrak{g})$-module, put $T_{w,\eta}(X) = S_{w,\eta} \otimes_{U(\mathfrak{g})} wX$.

Lemma

$T_{w,0}$ preserves the category $\mathcal{O}$.

$T_w := T_{w,0}$ is defined by Arkhipov and called twisting functor.

twisting functors (examples)

G: split, $M(\lambda)$: the Verma module with highest weight $\lambda - \rho$.
$M(\lambda)^*$: dual of $M(\lambda)$ in the category $\mathcal{O}$.
λ: dominant $\iff \langle \alpha, \lambda \rangle \geq 0$ for all positive root α.

- $T_e M = M$ (e: the unit element of W).
- $T_{w_0} M(\lambda) = M(w_0\lambda)^*$ (w_0: the longest element of W).
- $T_w M(\lambda) = M(w\lambda)$ if $-w\lambda$ is dominant
- $\mathrm{Ch}\, T_w M(\lambda) = \mathrm{Ch}\, M(w\lambda)$ (Ch: the character).
- If $\mathrm{Ch}\, M = \sum_{v \in W} c_v \,\mathrm{Ch}\, M(v\lambda)$ then $\sum_i (-1)^i \,\mathrm{Ch}\, L^i T_w M = \sum_{v \in M} c_v \,\mathrm{Ch}\, M(wv\lambda)$.

$T_w M(w^{-1}\lambda)$ is called the twisted Verma module.

module I_i'

Lemma

$I_i' \simeq T_{w_i,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)))$.

$$
\begin{aligned}
&I_i' \simeq \\
&U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{P}(w_i \overline{N} w_i^{-1} \cap N_0)\eta^{-1} \otimes J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))
\end{aligned}
$$

$$
\begin{aligned}
&T_{w_i,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))) \\
&= S_{w,\eta} \otimes_{U(\mathrm{Ad}(w_i)\mathfrak{p})} w_i J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))) \\
&= U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \left(\sum_{k_s \in \mathbb{Z}_{\geq 0}} (e_1 - \eta(e_1))^{-(k_1+1)} \dots (e_l - \eta(e_l))^{-(k_l+1)} \right) \\
&\qquad \otimes_{\mathbb{C}} J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)))
\end{aligned}
$$

module I_i' (2)

$e_1, \ldots, e_l$: basis of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$.
$x_s\colon w_i\overline{N}w_i^{-1} \cap N_0 \to \mathbb{C}$, $\exp(a_1e_1)\cdots\exp(a_le_l) \mapsto a_s$: polynomial
Define $I_i' \to T_{w_i,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda+\rho)))$ by

$$T \otimes ((-1)^{k_1}k_1!x_1^{k_1})\ldots((-1)^{k_l}k_l!x_l^{k_l})\eta^{-1} \otimes u \\ \mapsto T \otimes (e_1-\eta(e_1))^{-(k_1+1)}\ldots(e_l-\eta(e_l))^{-(k_l+1)} \otimes u.$$

This map gives isomorphism as a $\mathbb{C}$-vector space (obvious!).

Lemma

This map is a $U(\mathfrak{g})$-module homomorphism.

The first theorem is proved.

module $J_\eta^*(I(\sigma,\lambda))$

Proposition

$$J_\eta^*(V) = \Gamma_\eta(C(J^*(V))).$$

$\operatorname{supp}\eta := \{\alpha\colon \text{simple root} \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}$.

- $\operatorname{supp}\eta = \emptyset$: obvious.
- $\operatorname{supp}\eta = \{\text{simple root}\}$ (non-degenerate): the result of Matumoto.

This proposition implies the second main theorem since $\Gamma_\eta \circ C$ is exact.

proof of the Proposition

Proof.

$\mathfrak{p}_\eta = \mathfrak{m}_\eta \oplus \mathfrak{a}_\eta \oplus \mathfrak{n}_\eta$: the complexification of the parabolic subgroup corresponding to $\operatorname{supp}\eta$ and its Langlands decomposition.
$\eta_0 \colon U(\mathfrak{m}_\eta \cap \mathfrak{n}_0) \to \mathbb{C}$: restriction of η.

$$J^*_\eta(V) = \varinjlim_k (V/(\operatorname{Ker}\eta)^k V)^*$$
$$= \varinjlim_{k,l} ((V/\mathfrak{n}_\eta^k V)/(\operatorname{Ker}\eta_0)^l (V/\mathfrak{n}_\eta^k V))^*.$$

$V/\mathfrak{n}_\eta^k V$: Harish-Chandra module of $\mathfrak{m}_\eta$
$\rightsquigarrow$ We can apply Matumoto's result to $V/\mathfrak{n}_\eta^k V$. □

Whittaker vectors

V: $\mathfrak{g}$-module

$$\mathrm{Wh}_\eta(V) := \{v \in V \mid (\mathrm{Ker}\,\eta)v = 0\} : (\text{Whittaker vectors}).$$

V: a finite-length representation of G.

$$\mathrm{Wh}_\eta^\infty(V) := \mathrm{Wh}_\eta(V'),$$
$$\mathrm{Wh}_\eta^*(V) := \mathrm{Wh}_\eta((V_{K\text{-finite}})^*).$$

$\longleftrightarrow$ Whittaker models

To determine $\mathrm{Wh}_\eta^\blacktriangle$,

1. determine $\mathrm{Wh}_\eta(I_i/I_{i-1})$.
2. analyze $0 \to \mathrm{Wh}_\eta(I_{i-1}) \to \mathrm{Wh}_\eta(I_i) \to \mathrm{Wh}_\eta(I_i/I_{i-1})$ (exact).

non-degenerate case (well-known)

Assume $\operatorname{supp}\eta = \{\text{simple root}\}$.

- Wh_η^∞:
  if $i \neq r$ (i.e., w_i is not maximal) then $I_i/I_{i-1} = 0$
  $\Rightarrow J'_\eta(I(\sigma,\lambda)) = T_{w_r,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_r^{-1}\eta}(\sigma \otimes (\lambda+\rho))$.
  $\Rightarrow \dim \mathrm{Wh}_\eta^\infty(I(\sigma,\lambda)) = \dim J'_{w_r^{-1}\eta}(\sigma \otimes (\lambda+\rho))$.
- Wh_η^*:
  By the result of Kostan-Lynch, $V \mapsto \mathrm{Wh}_\eta(C(V))$ $(V \in \mathcal{O})$ is exact.
  $\Rightarrow \dim \mathrm{Wh}_\eta^*(I(\sigma,\lambda))$ is determined by
  $\mathrm{Ch}\, J^*(I(\sigma,\lambda)) = \sum_i \mathrm{Ch}(I_i/I_{i-1})$
  $\Rightarrow$ We can determine $\dim \mathrm{Wh}_\eta(J_\eta^*(I(\sigma,\lambda)))$.

generic case

Everything become easy if an infinitesimal character is far from integral.

1. The exact sequence $0 \to I_{i-1} \to I_i \to I_i/I_{i-1} \to 0$ splits.
2. All the Whittaker vectors of $T_{w_i,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))$ or $\Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes (\lambda + \rho)))))$ "come from" $J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))$ or $J^*_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))$.

generic case: Wh_η^∞

- $u \in I_i' = I_i/I_{i-1}$. The integral defining u is holomorphic at λ.
  $\Rightarrow$ This gives the lift of u on I_i.
  $\Rightarrow$ The exact sequence $0 \to I_{i-1} \to I_i \to I_i/I_{i-1} \to 0$ splits.
- $\mathrm{Wh}_\eta(I_i/I_{i-1}) = \mathrm{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(H^0(\mathfrak{n}_\eta, I_i/I_{i-1}))$.
  By the $\mathfrak{a}_\eta$-weight and the Harish-Chandra isomorphism, we can deduce the problem to determine $\dim \mathrm{Wh}_\eta(I_i/I_{i-1})$ to the subalgebra $\mathfrak{m}_\eta$, i.e., the non-degenerate case.

generic case: Wh_η^*

- $\{I_i\}$: the filtration of $J^*(I(\sigma, \lambda)) = J'(I(\sigma, \lambda))$.
  $\Rightarrow 0 \to I_{i-1} \to I_i \to I_i/I_{i-1} \to 0$: splits (by the theory of category $\mathcal{O}$)
  $\Rightarrow 0 \to \widetilde{I_{I-1}} \to \widetilde{I_i} \to \widetilde{I_i}/\widetilde{I_{i-1}} \to 0$: splits
- determining $\mathrm{Wh}_\eta(\widetilde{I_i}/\widetilde{I_{i-1}})$: We can use the same method as in the case of Wh_η^∞.

dimension of the Whittaker vectors

Theorem

If λ is generic then

1. $$\dim \mathrm{Wh}_\eta^\infty(I(\sigma,\lambda)) = \sum_{w\in W(M),\ \eta|_{wNw^{-1}\cap N_0}=1} \dim \mathrm{Wh}_\eta^\infty(\sigma).$$

2. $$\dim \mathrm{Wh}_\eta^*(I(\sigma,\lambda)) = \sum_{w\in W(M)} \dim \mathrm{Wh}_{w^{-1}\eta}^*(\sigma).$$

When σ is finite-dimensional, this is Oshima's results.