Functors connecting graded Hecke algebras and real reductive Lie groups

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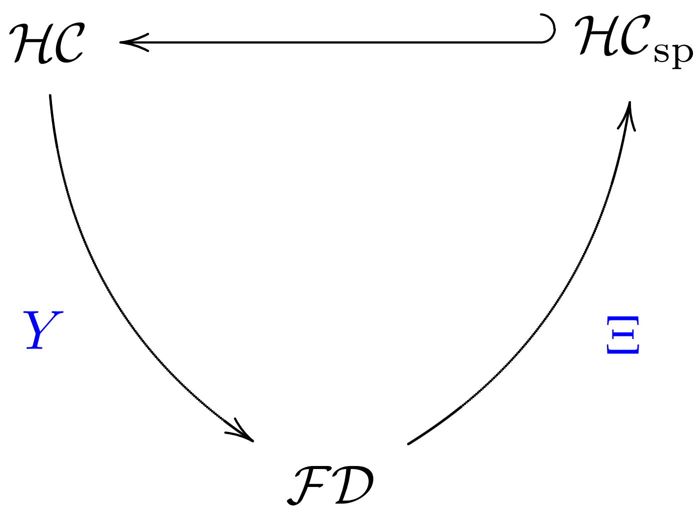
[a]Some mistakes are fixed after the talk.

Introduction

$\mathcal{HC} = \mathcal{HC}(G)$: the categoty of Harish-Chandra modules

$\mathcal{HC}_{\mathrm{sp}} = \mathcal{HC}_{\mathrm{sp}}(G)$: the categoty of Harish-Chandra modules that are generated by single-petaled K-types ($\mathcal{HC}_{\mathrm{sp}}$ is not an Abelian category nor an exact category!),

$\mathcal{FD} = \mathcal{FD}(\mathbf{H})$: the categoty of finite-dimensional $\mathbf{H}$-modules,



Real reductive Lie group

$G = KAN$: a real reductive group in the Harish-Chandra class,

$\mathfrak{g}_{\mathbb{R}}$: Lie algebra of G,

$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$: complexified Lie algebras,

θ : involution such that $K = G^{\theta}$,

$B(\cdot, \cdot)$: nondegenerate, invariant, symmetric bilinear form such that $-B(\cdot, \theta\cdot)$ is positive definite on $\mathfrak{g}_{\mathbb{R}}$,

$\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$: system of restricted roots,

$M := Z_K(A)$,

$W = W(G, A) = N_K(A)/M$: Weyl group,

$\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \sum_{\alpha \in \Sigma} \mathfrak{g}_{\alpha}$: root space decomposition,

Σ^{+} : positive roots cooresponding to $\mathfrak{n}$,

$\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^{+}} (\dim \mathfrak{g}_{\alpha}) \alpha \in \mathfrak{a}^{*}$,

The 0th $\mathfrak{n}$-homology

For a $(\mathfrak{g}, K)$-module $\mathscr{Y}$, its 0th $\mathfrak{n}$-homology is by definition

$$H_0(\mathfrak{n}, \mathscr{Y}) := \mathscr{Y} / \mathfrak{n}\mathscr{Y},$$

which is an $(\mathfrak{m} + \mathfrak{a}, M)$-module. There is a natural map

$$\mathscr{Y} \to H_0(\mathfrak{n}, \mathscr{Y})^M.$$

Example. For $V \in \widehat{K}$,

$$\begin{aligned} H_0\Big(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\Big)^M &= \Big(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V \,/\, \mathfrak{n}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\Big)^M \\ &= \Big(U(\mathfrak{n}+\mathfrak{a}) \otimes_{\mathbb{C}} V \,/\, \mathfrak{n}\, U(\mathfrak{n}+\mathfrak{a}) \otimes_{\mathbb{C}} V\Big)^M \\ &= \Big(U(\mathfrak{a}) \otimes_{\mathbb{C}} V\Big)^M \\ &= S(\mathfrak{a}) \otimes_{\mathbb{C}} V^M. \end{aligned}$$

Theorem (Casselman, Osborne). $0 < \dim H_0(\mathfrak{n}, \mathscr{Y}) < \infty$ for $\mathscr{Y} (\neq 0) \in \mathcal{HC}$.

Graded Hecke algebra

Π : simple roots.

For each indivisible root $\alpha \in \Sigma \setminus 2\Sigma$, put

$$\mathbf{m}_\alpha := \dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha}.$$

The graded Hecke algebra $\mathbf{H}$ for $(\mathfrak{a}, \Pi, \mathbf{m}_\alpha)$ is a unique $\mathbb{C}$-algebra satisfying

(i) as a linear space $\mathbf{H} = S(\mathfrak{a}) \otimes \mathbb{C}W$,

(ii) $S(\mathfrak{a}) \xrightarrow{\sim} S(\mathfrak{a}) \otimes 1$, $\mathbb{C}W \xrightarrow{\sim} 1 \otimes \mathbb{C}W$ are subalgebras,

(iii) $\xi \cdot s_\alpha = s_\alpha \cdot s_\alpha(\xi) - \mathbf{m}_\alpha \, \alpha(\xi)$ for $\xi \in \mathfrak{a}$ and $\alpha \in \Pi$
where $s_\alpha \in W$ is the reflection in α.

Generalized Harish-Chandra homomorphisms

Suppose $V \in \widehat{K}$. Then V^M is naturally a W-module.

The generalized H-C homomorphism $\gamma_V : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V \to \mathbf{H} \otimes_{\mathbb{C}W} V^M$ is the composition of the natural map

$$(*) \qquad U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V \longrightarrow H_0\big(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\big)^M$$

and the following isomorphisms

$$\begin{aligned} H_0\big(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\big)^M &\simeq S(\mathfrak{a}) \otimes V^M \xrightarrow[-\rho\text{-shift for the left factor}]{\sim} S(\mathfrak{a}) \otimes V^M \\ &\simeq S(\mathfrak{a}) \otimes \mathbb{C}W \otimes_{\mathbb{C}W} V^M \simeq \mathbf{H} \otimes_{\mathbb{C}W} V^M. \end{aligned}$$

We often consider γ_V is just the map $(*)$ and the latter part merely defines the canonical $\mathbf{H}$-module structure for $H_0\big(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\big)^M$.

If $V = \mathbb{C}$ (the trivial K-type) then

$$\begin{array}{ccccc} U(\mathfrak{g}) & \twoheadrightarrow & U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} & & \\ & & \downarrow \simeq & & \\ & & U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C} & \xrightarrow{\gamma_{\mathbb{C}}} & \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C}^M \\ & & & & \| \\ & & & & \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C} \xrightarrow{\simeq} S(\mathfrak{a}) \end{array}$$

reduces to the classical Harish-Chandra homomorphism for G/K

$$\gamma : U(\mathfrak{g}) = \big(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}\big) \oplus U(\mathfrak{a}) \xrightarrow{\text{proj.}} U(\mathfrak{a}) = S(\mathfrak{a}) \xrightarrow{-\rho\text{-shift}} S(\mathfrak{a}).$$

So we have the isomorphism

$$\mathrm{Hom}_K\big(\mathbb{C}, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C}\big) \xrightarrow{\sim} \mathrm{Hom}_W\big(\mathbb{C}^M, \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C}^M\big) \; ; \; \Phi \mapsto \gamma_{\mathbb{C}} \circ \Phi|_{\mathbb{C}^M} \; (= \gamma_{\mathbb{C}} \circ \Phi).$$

Single-petaled K-types

For each $\alpha \in \Sigma$ fix a root vector $X_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{R}$ so that $-B(X_\alpha, \theta X_\alpha) = 2/|\alpha|^2$ and put $Z_\alpha = \sqrt{-1}(X_\alpha + \theta X_\alpha)$.

Definition. We call a K-type $V \in \widehat{K}$ is single-petaled if

$$V^M \neq \{0\} \quad \text{and} \quad Z_\alpha(Z_\alpha^2 - 4)V^M = \{0\} \ (\forall \alpha \in \Sigma).$$

The collection of single-petaled K-types is denoted by $\widehat{K}_{\mathrm{sp}}$.

Example.

(i) The trivial K-type $\mathbb{C}$ is single-petaled.
Each K-type appearing in $\mathfrak{s} := \mathfrak{g}^{-\theta}$ is single-petaled.

(ii) For a complex G, Broer's smallness $\Leftrightarrow$ to be single-petaled.

(iii) For a split G, Barbasch's petiteness $\Rightarrow$ to be single-petaled.

Key lemma

Lemma 1. Suppose $E, V \in \widehat{K}_{\mathrm{sp}}$. If $\Psi \in \mathrm{Hom}_K\big(E, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\big)$ then the linear map

$$E^M \hookrightarrow E \xrightarrow{\Psi} U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V \xrightarrow{\gamma_V} \mathbf{H} \otimes_{\mathbb{C}W} V^M$$

is a W-homomorphism. Namely, this correspondence defines a map

$$\Gamma_V^E : \mathrm{Hom}_K\big(E, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V\big) \to \mathrm{Hom}_W\big(E^M, \mathbf{H} \otimes_{\mathbb{C}W} V^M\big).$$

Remark.

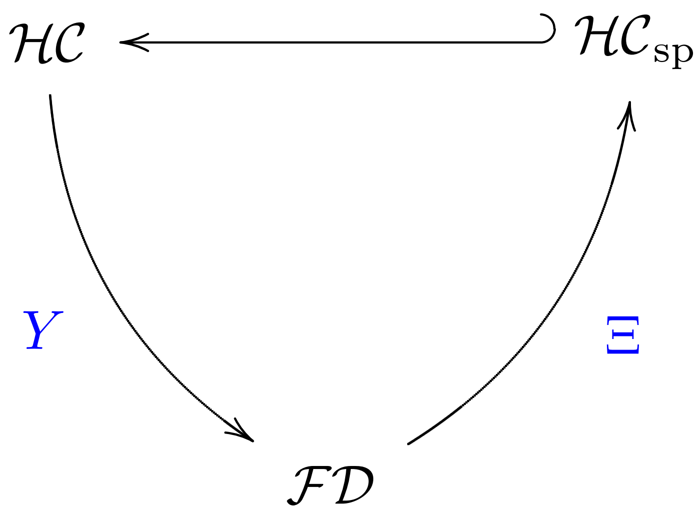
(i) The natural identification $\mathbf{H} \otimes_{\mathbb{C}W} V^M \simeq S(\mathfrak{a}) \otimes V^M$ does not commute with W-actions.

(ii) In general, Γ_V^E is neither injective nor surjective. But if at least E or V is trivial then Γ_V^E is an isomorphism.

Functors in this talk

$\mathcal{HC}_{\mathrm{sp}} = \mathcal{HC}_{\mathrm{sp}}(G)$: the categoty of Harish-Chandra modules that are generated by K-types in $\widehat{K}_{\mathrm{sp}}$ ($\mathcal{HC}_{\mathrm{sp}}$ is not an Abelian category!),

$\mathcal{FD} = \mathcal{FD}(\mathbf{H})$: the categoty of finite-dimensional $\mathbf{H}$-modules,



Definition of the functor $\Xi : \mathcal{FD} \to \mathcal{HC}_{\mathrm{sp}}$

$P := MAN$ (minimal parabolic).

Suppose $(\sigma, \mathscr{X}) \in \mathcal{FD}(\mathbf{H})$.
$\rightsquigarrow$ σ extends to a P-action by $\sigma(man) = e^{\sigma(\log a)} \in \operatorname{End} \mathscr{X}$.

$$\operatorname{Ind}_P^G \mathscr{X} := \Big\{ f : G \xrightarrow{C^\infty} \mathscr{X};\ f(gman) = e^{-(\sigma+\rho)(\log a)} f(g) \Big\}.$$

For any $V \in \widehat{K}_{\mathrm{sp}}$, the Frobenius reciprocity gives an isomorphism

$$\operatorname{Hom}_K(V, \operatorname{Ind}_P^G \mathscr{X}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}}(V^M, \mathscr{X}) \ ; \quad \Phi \mapsto \big(v \mapsto \Phi[v](e)\big).$$

$$\rightsquigarrow \quad V \otimes \operatorname{Hom}_W(V^M, \mathscr{X}) \subset V \otimes \operatorname{Hom}_{\mathbb{C}}(V^M, \mathscr{X})$$
$$\simeq \underbrace{V \otimes \operatorname{Hom}_K(V, \operatorname{Ind}_P^G \mathscr{X})}_{V\text{-isotypic comp. of } \operatorname{Ind}_P^G \mathscr{X}} \subset \operatorname{Ind}_P^G \mathscr{X}.$$

$\Xi(\mathscr{X}) :=$ the $(\mathfrak{g}, K)$-sub of $\operatorname{Ind}_P^G \mathscr{X}$ spanned by $V \otimes \operatorname{Hom}_W(V^M, \mathscr{X})$ $(V \in \widehat{K}_{\mathrm{sp}})$
$\in \mathcal{HC}_{\mathrm{sp}}$.

$$\mathrm{Hom}_K(V, \mathrm{Ind}_P^G \mathscr{X}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V^M, \mathscr{X}) \ ; \quad \Phi \mapsto \mathrm{ev} \circ \Phi|_{V^M}$$

where ev is the evaluation at $e \in G$.

$V \otimes \mathrm{Hom}_W(V^M, \mathscr{X}) \subset V \otimes \mathrm{Hom}_K(V, \mathrm{Ind}_P^G \mathscr{X}) \subset \mathrm{Ind}_P^G \mathscr{X}$.

$\Xi(\mathscr{X}) :=$ the $(\mathfrak{g}, K)$-sub of $\mathrm{Ind}_P^G \mathscr{X}$ spanned by $V \otimes \mathrm{Hom}_W(V^M, \mathscr{X})$ $(V \in \widehat{K}_{\mathrm{sp}})$.

$$\begin{array}{ccccc}
\bigoplus_{V \in \widehat{K}_{\mathrm{sp}}} U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V \otimes \mathrm{Hom}_W(V^M, \mathscr{X}) & \longrightarrow\!\!\!\!\rightarrow & \Xi(\mathscr{X}) & & (\mathfrak{g}, K)\text{-homo} \\
\Big\downarrow \bigoplus \gamma_V \otimes \mathrm{id} & \circlearrowleft & \Big\downarrow \mathrm{ev} & & \\
\bigoplus_{V \in \widehat{K}_{\mathrm{sp}}} \mathbf{H} \otimes_{\mathbb{C}W} V^M \otimes \mathrm{Hom}_W(V^M, \mathscr{X}) & \xrightarrow{\text{natural map}} & \mathscr{X} & & \mathbf{H}\text{-homo}
\end{array}$$

The above is easily checked. Hence for any $E \in \widehat{K}_{\mathrm{sp}}$, Lemma 1 implies

$$\begin{array}{ccc}
\mathrm{Hom}_K\left(E, \bigoplus_{V \in \widehat{K}_{\mathrm{sp}}} U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V \otimes \mathrm{Hom}_W(V^M, \mathscr{X})\right) & \longrightarrow\!\!\!\!\rightarrow & \mathrm{Hom}_K(E, \Xi(\mathscr{X})) \\
\Big\downarrow \bigoplus \Gamma_V^E \otimes \mathrm{id} & \circlearrowleft & \Big\downarrow \mathrm{ev} \circ \blacksquare|_{E^M} \\
\mathrm{Hom}_W\left(E^M, \bigoplus_{V \in \widehat{K}_{\mathrm{sp}}} \mathbf{H} \otimes_{\mathbb{C}W} V^M \otimes \mathrm{Hom}_W(V^M, \mathscr{X})\right) & \longrightarrow & \mathrm{Hom}_W(E^M, \mathscr{X})
\end{array}$$

Correspondence of multiplicities

Theorem. For $\mathscr{X} \in \mathcal{FD}$ and $E \in \widehat{K}_{\mathrm{sp}}$ we have

$$\begin{array}{ccc} \operatorname{Hom}_K(E, \operatorname{Ind}_P^G \mathscr{X}) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathbb{C}}(E^M, \mathscr{X}) \ ; \ \Phi \mapsto \operatorname{ev} \circ \Phi|_{E^M} \\ \uparrow & & \uparrow \\ \operatorname{Hom}_K(E, \Xi(\mathscr{X})) & \xrightarrow{\sim} & \operatorname{Hom}_W(E^M, \mathscr{X}) \end{array}$$

In particular, $[E : \Xi(\mathscr{X})] = \dim \operatorname{Hom}_W(E^M, \mathscr{X})$.

Remark.

It is trivial that the image of $\operatorname{Hom}_K(E, \Xi(\mathscr{X}))$ contains $\operatorname{Hom}_W(E^M, \mathscr{X})$ since its preimage is a part of generators of $\Xi(\mathscr{X})$.

The point is the image stays in $\operatorname{Hom}_W(E^M, \mathscr{X})$ after taking a $U(\mathfrak{g})$-span of generators.

Correspondence of parabolic inductions

For $\Theta \subset \Pi$ define

$P_\Theta := M_\Theta A_\Theta N_\Theta = G_\Theta N_\Theta$: standard parabolic,

$\mathbf{H}_\Theta := S(\mathfrak{a}) \otimes \mathbb{C}W_\Theta \subset \mathbf{H}$ with $W_\Theta := \langle s_\alpha;\, \alpha \in \Theta \rangle$,

$\Xi_\Theta : \mathcal{FD}(\mathbf{H}_\Theta) \to \mathcal{HC}(G_\Theta)$: Ξ for $(G_\Theta, \mathbf{H}_\Theta)$.

Definition. Each $\mathscr{X}_\Theta \in \mathcal{FD}(\mathbf{H}_\Theta)$ (co-)induces an $\mathbf{H}$-module

$$\mathrm{Ind}_{\mathbf{H}_\Theta}^{\mathbf{H}} \mathscr{X}_\Theta := \left\{ f \in \mathrm{Hom}_{\mathbb{C}}(\mathbf{H}, \mathscr{X}_\Theta)\,;\; f(h_\Theta \cdot) = h_\Theta f(\cdot) \text{ for } (h_\Theta \in \mathbf{H}_\Theta) \right\},$$

on which $\mathbf{H}$ acts from the right.

Theorem. For $\mathscr{X}_\Theta \in \mathcal{FD}(\mathbf{H}_\Theta)$ there is a $(\mathfrak{g}, K)$-homo

$$\Xi\Big(\mathrm{Ind}_{\mathbf{H}_\Theta}^{\mathbf{H}} \mathscr{X}_\Theta\Big) \xrightarrow{\quad \beta_{\mathscr{X}_\Theta} \quad} \mathrm{Ind}_{G_\Theta N_\Theta}^{G} \Big(\Xi_\Theta(\mathscr{X}_\Theta) \boxtimes \mathbb{C}\Big)\Big|_K$$

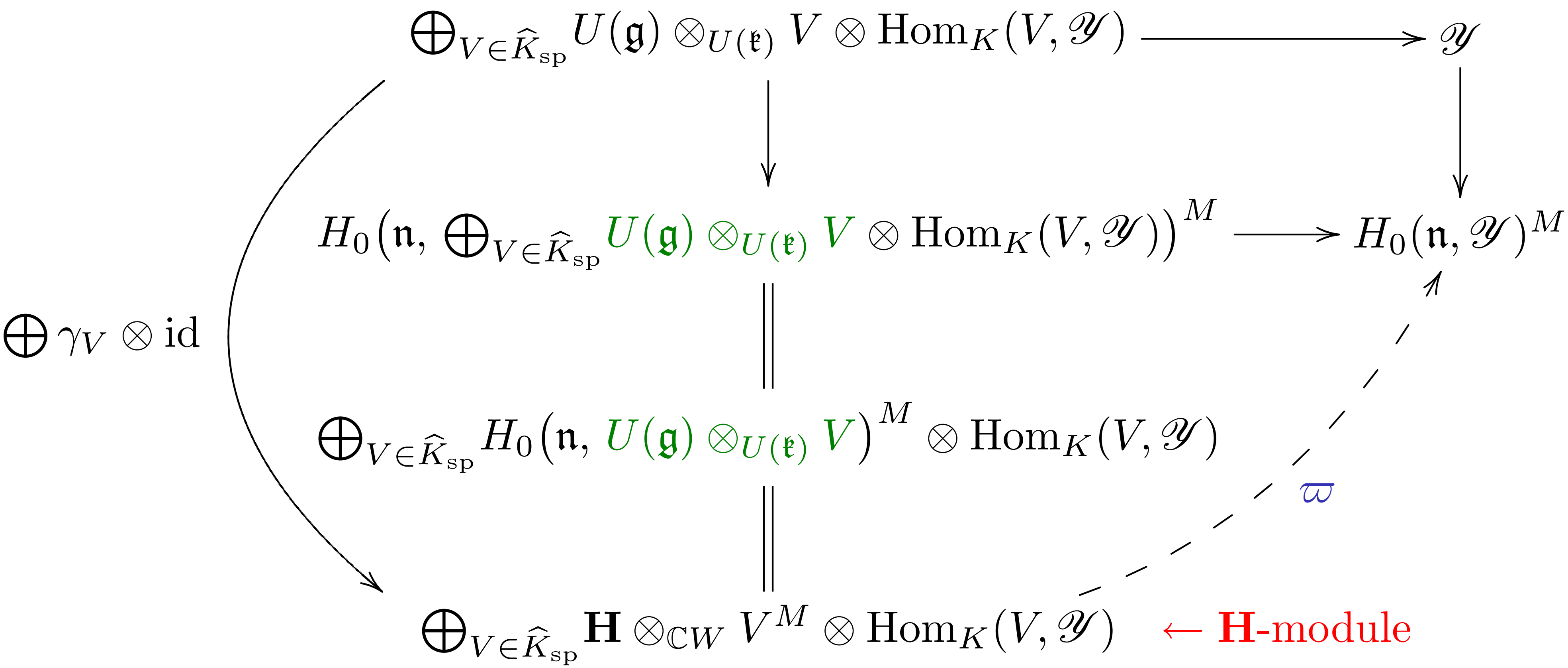
induction in $\mathbf{H}$ induction in G

such that for any $V \in \widehat{K}_{\mathrm{sp}}$ it induces

$$\mathrm{Hom}_K\Big(V, \Xi\Big(\mathrm{Ind}_{\mathbf{H}_\Theta}^{\mathbf{H}} \mathscr{X}_\Theta\Big)\Big) \xrightarrow{\sim} \mathrm{Hom}_K\Big(V, \mathrm{Ind}_{G_\Theta N_\Theta}^{G} \Big(\Xi_\Theta(\mathscr{X}_\Theta) \boxtimes \mathbb{C}\Big)\Big|_K\Big).$$

Definition of the functor $Y : \mathcal{HC} \to \mathcal{FD}$

For a given $\mathscr{Y} \in \mathcal{HC}$ consider the following commutative diagram:



$$Y(\mathscr{Y}) := \bigoplus_{V \in \widehat{K}_{\mathrm{sp}}} \mathbf{H} \otimes_{\mathbb{C}W} V^M \otimes \mathrm{Hom}_K(V, \mathscr{Y}) \Big/ \mathbf{H}\, \mathrm{Ker}\, \varpi \in \mathcal{FD}$$

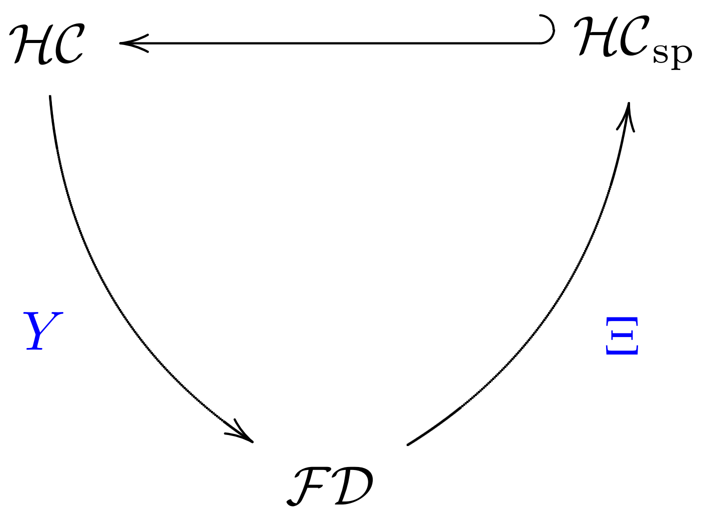
(Note $\dim Y(\mathscr{Y}) \leq \dim H_0(\mathfrak{n}, \mathscr{Y})^M < \infty$.)

Relation between Y and Ξ

Theorem. $Y : \mathcal{HC}_{\mathrm{sp}} \to \mathcal{FD}$ is the left adjoint functor of $\Xi : \mathcal{FD} \to \mathcal{HC}_{\mathrm{sp}}$.

Namely, for any $\mathscr{X} \in \mathcal{FD}$ and $\mathscr{Y} \in \mathcal{HC}_{\mathrm{sp}}$

$$\mathrm{Hom}_{\mathbf{H}}\big(Y(\mathscr{Y}), \mathscr{X}\big) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}\big(\mathscr{Y}, \Xi(\mathscr{X})\big).$$



Are they exact?

Proposition. For an exact sequence in $\mathcal{FD}$

$$0 \to \mathscr{X}_1 \to \mathscr{X}_2 \to \mathscr{X}_3 \to 0,$$

the following are exact in $\mathcal{HC}$ (not $\mathcal{HC}_{\mathrm{sp}}$!)

$$0 \to \Xi(\mathscr{X}_1) \to \Xi(\mathscr{X}_2), \qquad \Xi(\mathscr{X}_2) \to \Xi(\mathscr{X}_3) \to 0.$$

Proposition. For an exact sequence in $\mathcal{HC}$

$$\mathscr{Y}_1 \to \mathscr{Y}_2 \to \mathscr{Y}_3 \to 0,$$

the following is exact in $\mathcal{FD}$

$$Y(\mathscr{Y}_2) \to Y(\mathscr{Y}_3) \to 0.$$

Moreover, if $\mathscr{Y}_1 \in \mathcal{HC}_{\mathrm{sp}}$ then the following is exact in $\mathcal{FD}$

$$Y(\mathscr{Y}_1) \to Y(\mathscr{Y}_2) \to Y(\mathscr{Y}_3) \to 0.$$

Example 1 : spherical principal series

For $\lambda \in \mathfrak{a}^*$ put

$$P_G^\lambda := \mathrm{Ind}_{MAN}^G(\mathbb{C} \boxtimes \mathbb{C}_\lambda \boxtimes \mathbb{C})\big|_K, \quad P_{\mathbf{H}}^\lambda := \mathrm{Ind}_{S(\mathfrak{a})}^{\mathbf{H}} \mathbb{C}_\lambda.$$

Theorem.

If P_G^λ has a cyclic K-fixed vector, namely if for any $\alpha \in \Sigma_1 (= \Sigma \setminus \Sigma)$

$$(*) \qquad \begin{aligned} -\lambda(\alpha^\vee) \neq \dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha}, \dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha} + 4, \dots, \\ \dim \mathfrak{g}_\alpha + 2, \dim \mathfrak{g}_\alpha + 6, \dim \mathfrak{g}_\alpha + 10, \dots, \end{aligned}$$

then

$$\Xi(P_{\mathbf{H}}^\lambda) = P_G^\lambda, \qquad Y(P_G^\lambda) = P_{\mathbf{H}}^\lambda.$$

Corollary.

If $\lambda(\alpha^\vee) \neq \pm(\dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha})$ for $\forall \alpha \in \Sigma_1$, we can take $w \in W$ so that

$$\Xi(P_{\mathbf{H}}^\lambda) = P_G^{w\lambda}.$$

Example 2 : the case $G = SL(2,\mathbb{R})$

Identify $\lambda \in \mathfrak{a}^*$ with a complex number $\lambda(\alpha^\vee)$.

$$\begin{cases} P_G^{+,\lambda} = \mathrm{Ind}_{MAN}^G(\mathbb{C}_+ \boxtimes \mathbb{C}_\lambda \boxtimes \mathbb{C})\big|_K, \quad P_G^{-,\lambda} = \mathrm{Ind}_{MAN}^G(\mathbb{C}_- \boxtimes \mathbb{C}_\lambda \boxtimes \mathbb{C})\big|_K, \\ D_n^{\pm} = \text{(limit of) discrete series } (n = 0, 1, \ldots), \\ F_n = \text{irreducible representation with dimension } n = 1, 2, \ldots. \end{cases}$$

$$\left\{ P_{\mathbf{H}}^{\lambda} = \mathrm{Ind}_{S(\mathfrak{a})}^{\mathbf{H}} \mathbb{C}_\lambda, \quad \mathbb{C}_{-\rho,\mathrm{triv}}, \quad \mathbb{C}_{\rho,\mathrm{sgn}}. \right.$$

Theorem. (i). $\Xi : \mathcal{FD} \to \mathcal{HC}$ is an exact functor. $Y : \mathcal{HC} \to \mathcal{FD}$ maps each simple object to a simple object or 0.

(ii).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}_{\rho,\mathrm{sgn}} & \longrightarrow & P_{\mathbf{H}}^{1} & \longrightarrow & \mathbb{C}_{-\rho,\mathrm{triv}} & \longrightarrow & 0 \\ \Xi & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & D_1^+ \oplus D_1^- & \longrightarrow & P_G^{+,1} & \longrightarrow & F_1 & \longrightarrow & 0 \\ Y & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ & & \mathbb{C}_{\rho,\mathrm{sgn}} \oplus \mathbb{C}_{\rho,\mathrm{sgn}} & \longrightarrow & P_{\mathbf{H}}^{1} & \longrightarrow & \mathbb{C}_{-\rho,\mathrm{triv}} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}_{-\rho,\mathrm{triv}} & \longrightarrow & P_{\mathbf{H}}^{-1} & \longrightarrow & \mathbb{C}_{\rho,\mathrm{sgn}} & \longrightarrow & 0 \\ \Xi & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & P_G^{+,-1} & \longrightarrow & D_1^+ \oplus D_1^- & \longrightarrow & 0 \\ Y & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathbb{C}_{-\rho,\mathrm{triv}} & \longrightarrow & P_{\mathbf{H}}^{-1} \oplus \mathbb{C}_{\rho,\mathrm{sgn}} & \longrightarrow & \mathbb{C}_{\rho,\mathrm{sgn}} \oplus \mathbb{C}_{\rho,\mathrm{sgn}} & \longrightarrow & 0 \end{array}$$

(iii). If $n = 3, 5, 7, \dots$ then $P^n_{\mathbf{H}} \simeq P^{-n}_{\mathbf{H}}$ is simple while $P_G^{+,\lambda}$, $P_G^{+,-\lambda}$ are not simple.

$$\begin{array}{ccccccccc} & & & & P^n_{\mathbf{H}} \simeq P^{-n}_{\mathbf{H}} : \text{simple} & & & & \\ & & & \Xi & \wr\downarrow & & & & \\ 0 & \longrightarrow & D_n^+ \oplus D_n^- & \longrightarrow & P_G^{+,n} & \longrightarrow & F_n & \longrightarrow & 0 \\ & Y & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ & & 0 & \longrightarrow & P^n_{\mathbf{H}} & \longrightarrow & 0 & & \text{(not exact)} \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_n & \longrightarrow & P_G^{+,-n} & \longrightarrow & D_n^+ \oplus D_n^- & \longrightarrow & 0 \\ & Y & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \text{(exact)} \end{array}$$

(iv). If λ is not an odd integer then $P_G^{+,\lambda} \simeq P_G^{+,-\lambda}$ and $P^{\lambda}_{\mathbf{H}} \simeq P^{-\lambda}_{\mathbf{H}}$ are simple and

$$\Xi\big(P^{\lambda}_{\mathbf{H}}\big) = P_G^{+,\lambda}, \qquad Y\big(P_G^{+,\lambda}\big) = P^{\lambda}_{\mathbf{H}}.$$

(v). Any constituent in $P_G^{-,\lambda} (\lambda \in \mathbb{C})$ has no relation with our functors. Namely, such constituent never appears as a factor of $\Xi(\mathscr{X})$ for any $\mathscr{X}$, and if $\mathscr{Y} \subset P_G^{-,\lambda}$ then $Y(\mathscr{Y}) = 0$.

Final remarks

(i) In the definition of Ξ and Y, we can replace $\widehat{K}_{\mathrm{sp}}$ with any $S \subset \widehat{K}_{\mathrm{sp}}$. Even then, all theorems and statements (excluding those for examples) are still valid. (The correspondence of principal series needs $\mathbb{C}_{\mathrm{triv}} \in S$. The exactness of Ξ for $SL(2, \mathbb{R})$ needs $S = \emptyset$ or $\widehat{K}_{\mathrm{sp}}$.)

(ii) There is a notion of quasi-single-petaled K-types ($\rightarrow \widehat{K}_{\mathrm{qsp}} \supset \widehat{K}_{\mathrm{sp}}$) and we can easily generalize the definition of Ξ for $\widehat{K}_{\mathrm{qsp}}$ so that the exacteness would be better. But I cannot find out the corresponding generalization of Y.

(iii) Correspondence of Langlands classifications?

(iv) Relations with other similar functors by Arakawa-Suzuki, Etingof-, Chiubotaru-Trapa, . . . ?