Uniqueness of certain mixed models: a new descent method*

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1 Main result

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $G = \mathrm{GL}_6(\mathbb{K})$ and S be its Ginzburg-Rallis subgroup:

$$S = \left\{ \begin{bmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{bmatrix} \mid a \in \mathrm{GL}_2(\mathbb{K}) \right\}.$$

Define a character χ_S of S by

$$\chi_S\left(\begin{bmatrix} 1 & b & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \right) = \chi_{\mathbb{K}^\times}(\det(a))\, \psi_{\mathbb{K}}(\mathrm{tr}(b+c)),$$

where $\psi_{\mathbb{K}}$ is a nontrivial unitary character of $\mathbb{K}$, and $\chi_{\mathbb{K}^\times}$ is any character of $\mathbb{K}^\times$.

Theorem (uniqueness of Ginzburg-Rallis models)

Let V be an irreducible, admissible, smooth Fréchet representation of G of moderate growth. Then we have

$$\dim \operatorname{Hom}_S(V, \mathbb{C}_{\chi_S}) \leq 1.$$

Remark: smooth Fréchet representations of moderate growth = (canonical) smooth globalization of Harish-Chandra modules, by Casselman-Wallach.

Terminologies: Whittaker models, linear models, mixed models

2 Reduction through Gelfand-Kazhdan criterion

Denote by Δ the Casimir element (with respect to real trace form), viewed as a bi-invariant differential operator on G.

Theorem: Let f be a tempered generalized function on G, which is an eigenvector of Δ. If f satisfies

$$f(sx) = f(xs^\tau) = \chi_S(s) f(x), \quad \text{for all } s \in S,$$

then

$$f(x) = f(x^\tau).$$

3 Generalized functions and differential operators

Let M be a smooth manifold.

- $C^{-\infty}(M)$: the space of generalized functions on M.
- For a locally closed subset Z of M, denote

$$C^{-\infty}(M;Z) = \{f \in C^{-\infty}(U) |\ \mathrm{supp}(f) \subseteq Z\},$$

  where U is any open subset containing Z as a closed subset.
- For a differential operator D on M, denote

$$C^{-\infty}(M;D) = \{f \in C^{-\infty}(M) |\ Df = 0\}.$$

- Given Z and D, denote

$$C^{-\infty}(M;Z;D) = C^{-\infty}(M;Z) \cap C^{-\infty}(M;D).$$

Suppose that a Lie group H acts smoothly on a manifold M.

- For a character χ of H, denote $C^{-\infty}_{\chi}(M)$ the space of χ-equivariant generalized functions.
- If a locally closed subset Z of M is H stable, denote by $C^{-\infty}_{\chi}(M; Z)$ the space of all f in $C^{-\infty}(M; Z)$ which are χ-equivariant.
- Similar notations: $C^{-\infty}_{\chi}(M; D)$; $C^{-\infty}_{\chi}(M; Z; D)$.

If M is a Nash manifold (those which is modeled locally by semialgebraic sets in $\mathbb{R}^n$), then one may define a notion of **temperedness**.

- $C^{-\xi}(M)$: the space of tempered generalized functions on M.
- Similar notations such as $C^{-\xi}_{\chi}(M; Z; D)$.

4 Generalized functions on G

We consider G as a $G \times G$ manifold:

$$(g_1, g_2)x = g_1 x g_2^\tau.$$

- For the problem at hand, the relevant subgroup is $H = S \times S$, and the character of H is $\chi = \chi_S \boxtimes \chi_S$.
- We need to show that every generalized function in $C_\chi^{-\xi}(G; \Delta - \lambda)$ is τ-invariant, for any λ.
- Following Bruhat, we decompose G into P-P^τ double cosets:

$$G = \bigsqcup_R G_R$$

  where P is the parabolic subgroup of containing S, and try to understand such generalized functions by imposing additional support conditions with respect to double cosets.

5 Three ingredients of the proof

(a) Reduction to an H stable open submanifold G' of G consisting of four double cosets, using the **transversality** of certain vector fields to all G_R's outside G'. The technique is due to Shalika.

(b) A **descent** argument from G' to a smaller H stable open submanifold G'', such that the corresponding problem on G'' is reduced to the following linear model problems: the uniqueness of trilinear models for GL_2, and the multiplicity one property for the pair $(\mathrm{GL}_2, \mathrm{GL}_1)$. This argument relies on two geometric notions attached to submanifolds.

(c) Use of the oscillator representation to conclude the uniqueness of the two afore-mentioned linear models. The specific phenomenon is called **first occurrence**.

6 Reduction to G' by transversality

Lemma: Let D_1 be a differential operator on M of order $k \geq 1$, which is transversal to a submanifold Z of M. Let D_2 be a differential operator on M which is tangential to Z. Then

$$C^{-\infty}(M; Z; D_1 + D_2) = 0.$$

Using "transversality" outside G' and the above lemma, we have

Proposition: Let $f \in C_\chi^{-\infty}(G)$. If f is an eigenvector of Δ, and f vanishes on G', then $f = 0$.

Remark: this step cuts off certain small submanifolds of G (those which are outside G').

7 First occurrence

For Step (c), we appeal to the following

Lemma: Let E be a finite dimensional non-degenerate quadratic space over $\mathbb{K}$, and let the orthogonal group $\mathrm{O}(E)$ act on E^k diagonally, where k is a positive integer. If $k < \dim E$, and if a **tempered** generalized function f on E^k is $\mathrm{SO}(E)$-invariant, then f is $\mathrm{O}(E)$-invariant.

The above lemma may be stated as that the determinant character of $\mathrm{O}(E)$ does not occur in Howe duality correspondence of $(\mathrm{O}(E), \mathrm{Sp}(2k))$ if $k < \dim E$. In fact the determinant character occurs if and only if $k \geq \dim E$.

8 Some geometric notions

Unipotent χ-incompatibility

Definition: An H stable submanifold Z of M is said to be unipotently χ-incompatible if for every $z_0 \in Z$, there is a local H slice $\mathfrak{Z}$ of Z, containing z_0, and a smooth map $\phi : \mathfrak{Z} \to H$ such that the followings hold for all $z \in \mathfrak{Z}$:

(a) $\phi(z) \in \mathrm{Stab}_z$,

(b) $\chi(\phi(z)) \neq 1$, and

(c) the linear map

$$\mathrm{T}_z(M)/\,\mathrm{T}_z(Z) \to \mathrm{T}_z(M)/\,\mathrm{T}_z(Z)$$

induced by the action of $\phi(z)$ on M is unipotent.

Key Lemma: Let Z be an H stable submanifold of M which is unipotently χ-incompatible. Then $C_\chi^{-\infty}(M; Z) = 0$.

Metrical properness

Let M is be a pseudo Riemannian manifold.

Definition:

(a) A submanifold Z of M is said to be metrically proper if for all $z \in Z$, the tangent space $\mathrm{T}_z(Z)$ is contained in a proper nondegenerate subspace of $\mathrm{T}_z(M)$.

(b) A second order differential operator D is said to be of Laplacian type if for all $x \in M$, the principal symbol

$$\sigma_2(D)(x) = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m,$$

where $u_1, u_2, \cdots, u_m$ is a basis of the tangent space $\mathrm{T}_x(M)$, and $v_1, v_2, \cdots, v_m$ is the dual basis in $\mathrm{T}_x(M)$.

Note that a Laplacian type differential operator is transversal to any metrically proper submanifold, from its very definition.

Lemma: Let Z be a metrically proper submanifold of M, and let D be a Laplacian type differential operator on M. Then

$$C^{-\infty}(M; Z; D) = 0.$$

Remark: This is a form of uncertainty principle.

$\mathrm{U}_\chi\,\mathrm{M}$ **property**

Let H be a Lie group acting smoothly on a pseudo Riemannian manifold M, and let χ be a character on H.

Definition: We say that an H stable locally closed subset Z of M has $\mathrm{U}_\chi\,\mathrm{M}$ property if there is a finite filtration

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_k \supset Z_{k+1} = \emptyset$$

of Z by H stable closed subsets of Z such that each $Z_i \setminus Z_{i+1}$ is a submanifold of M which is either unipotently χ-incompatible or metrically proper in M.

Lemma: Let Z be an H stable closed subset of M having $\mathrm{U}_\chi\,\mathrm{M}$ property. Then for any differential operator D of Laplacian type, we have

$$C_\chi^{-\infty}(M; Z; D) = 0.$$

9 From G' to G'' by $\mathrm{U}_\chi\,\mathrm{M}$

We shall define an H stable open submanifold G'' of G' which has three submanifolds $M_2, M_3, \check{M}_3$ as local H slices.

Set

$$M_2 = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \{I_2\} \subset G,$$

which is stable under the subgroup

$$H_2 = \mathrm{GL}_2(\mathbb{K}) = \{(x, x^{-\tau}) \mid x \in \mathrm{GL}_2^{\Delta}(\mathbb{K})\} \subset H.$$

Set

$$M_3 = \left\{ \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in G \mid x_{33} \neq x_{44} \right\},$$

which is stable under a certain (non-reductive) subgroup H_3 of H.

Also define a symmetric counterpart of M_3, called $\check{M}_3$.

Let

$$G'' = HM_2 \cup HM_3 \cup H\check{M}_3 \subset G'.$$

Proposition: As an H manifold, $G' \setminus G''$ has $\mathrm{U}_\chi\,\mathrm{M}$ property. Consequently if $f \in C_\chi^{-\infty}(G')$ is an eigenvector of Δ, and f vanishes on G'', then $f = 0$.

What we need to do: determine $G' \setminus G''$ explicitly and perform orbit analysis painstakingly to show the $\mathrm{U}_\chi\,\mathrm{M}$ property (finding slices and stabilizers to check unipotent χ-incompatibility; finding tangent spaces to check metrical properness).

10 The smaller models or the descents

Proposition: Let $H_2 = \mathrm{GL}_2(\mathbb{K})$ act on $M_2 = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ by

$$g(x, y) = (gxg^{-1}, gyg^{-1}), \quad g \in \mathrm{GL}_2(\mathbb{K}),$$

Then any H_2-invariant tempered generalized function on M_2 is τ-invariant, where

$$\tau(x, y) = (x^{\tau}, y^{\tau}).$$

Proof: Extend the action of H_2 on M_2 to the larger space $\mathfrak{gl}_2(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K})$. It suffices to prove the same on $\mathfrak{gl}_2(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K})$, and in fact on $\mathfrak{sl}_2(\mathbb{K}) \times \mathfrak{sl}_2(\mathbb{K})$.

View $\mathfrak{sl}_2(\mathbb{K})$ as a three-dimensional quadratic space under the trace form. The action of H_2 yields the diagonal action of $\mathrm{SO}(\mathfrak{sl}_2(\mathbb{K}))$ on $\mathfrak{sl}_2(\mathbb{K}) \times \mathfrak{sl}_2(\mathbb{K})$. By first occurrence lemma, any H_2-invariant tempered generalized function must be $\mathrm{O}(\mathfrak{sl}_2(\mathbb{K}))$-invariant. $\square$

Remark: the above proposition gives the multiplicity one property for the pair $(\mathrm{GL}_2 \times \mathrm{GL}_2, \Delta\mathrm{GL}_2)$, or equivalently the uniqueness of trilinear models for GL_2.

Now for the H_3 action on M_3 (and similarly for $\check{M}_3$):

- It amounts to uniqueness of certain mixed models for GL_3.
- Further reduced to the multiplicity one property for the pair $(\mathrm{GL}_2, \mathrm{GL}_1)$, by using a form of $\mathrm{U}_\chi\,\mathrm{M}$ property.
- It also follows from first occurrence lemma, now applied to a two-dimensional quadratic space.

11 Final remarks

- Apart from the "ultimate" models, the descent process also presents other intermediate models such as certain mixed models for $\mathrm{GL}_4 \times \mathrm{GL}_2$.
- One can prove exactly the same result for $G = \mathrm{GL}_3(\mathbb{D})$, where $\mathbb{D}$ is the quaternion division algebra. This is the non-split case.
- One can give a quick proof of the uniqueness of Whittaker models, by using the notion of unipotent χ-incompatibility.