REPRESENTATIONS WITH SCALAR K-TYPES AND APPLICATIONS

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ABSTRACT. We discuss some results of Shimura on invariant differential operators and extend a folklore theorem about spherical representations to representations with scalar K-types. We then apply the result to obtain non-trivial isomorphisms of certain representations arising from local theta correspondence, many of which are unipotent in the sense of Vogan.

1. INTRODUCTION AND MAIN RESULTS

Let G be a connected noncompact semisimple Lie group with finite center, and K be a maximal compact subgroup. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebra of G and its subalgebra corresponding to K. Then we have a Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0,$$

where $\mathfrak{p}_0$ is a certain subspace of $\mathfrak{g}_0$. We denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, respectively. The universal enveloping algebras of $\mathfrak{g}$ (resp., $\mathfrak{k}$) will be denoted by $\mathcal{U}(\mathfrak{g})$ (resp., $\mathcal{U}(\mathfrak{k})$).

For a vector space W, denote by $S(W)$ the symmetric algebra over W, and $S_r(W)$ the subspace of $S(W)$ consisting of homogeneous elements of degree r, where $r \in \mathbb{Z}_{\geq 0}$. Write

$$S^r(W) = \sum_{0 \leq k \leq r} S_k(W).$$

Let $\mathcal{U}^r(\mathfrak{g})$ be the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by elements of the form $Z_1 \cdots Z_s$, with $Z_i \in \mathfrak{g}$, $s \leq r$. Recall that there is a $\mathbb{C}$-linear bijection ψ of $S(\mathfrak{g})$ onto $\mathcal{U}(\mathfrak{g})$, called the symmetrization map. We have $\psi(S^r(\mathfrak{p})) \subseteq \mathcal{U}^r(\mathfrak{g})$.

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Let V be a Harish-Chandra $(\mathfrak{g}, K)$-module. For $\rho \in \hat{K}$, let V_ρ be the ρ-isotypic component of V. Here and after, $\hat{K}$ denotes the set of equivalent classes of irreducible (unitary) representations of K.

Our first result is the following

Theorem 1.1. *Suppose that*

$$\psi(S^r(\mathfrak{g})^G) + \mathcal{U}(\mathfrak{g})\mathfrak{k} \supseteq \psi(S^r(\mathfrak{p})^K), \qquad r \in \mathbb{Z}_{\geq 0}$$

(true at least for all classical G), and $\rho \in \hat{K}$ is one-dimensional. For an irreducible Harish-Chandra module V with $V_\rho \neq 0$, the infinitesimal character of V determines V up to infinitesimal equivalence.

The key ingredient for the proof of this simple criterion is a certain surjectivity result of Shimura on invariant differential operators ([Sh]). For ρ the trivial representation, Helgason showed [He2] that this surjectivity holds precisely when G does not contain any simple factor of the following four exceptional types: $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{6(-24)}$, $\mathfrak{e}_{7(-25)}$, $\mathfrak{e}_{8(-24)}$. See §2 for the precise statements and other discussions. Therefore excluding these four exceptional cases, irreducible spherical representations are completely determined by their infinitesimal characters.

We next apply the above result in the context of local theta correspondence. As it turns out, theta lifts of one dimensional representations often have one-dimensional K-types, especially at the so-called point of first occurrence. As it is very easy to compute their infinitesimal characters, our results can be used to identify these lifts quickly.

To be more precise, consider the dual pair [H1]

$$(O(p, q), Sp(2n, \mathbb{R})) \subseteq Sp(2N, \mathbb{R}),$$

where $N = 2(p + q)n$. The theta lift from $O(p, q)$ to $Sp(2n, \mathbb{R})$ will be denoted by $\theta^{p,q}_{\to n}$.

Denote by $1\!\!1$ the trivial representation of any group. It should be clear from the context which is the group concerned. Recall that a compact orthogonal group has two characters, the trivial and the determinant characters, while for a non-compact orthogonal group $O(k, l)$, there are four characters. They are denoted by $1\!\!1$, det, $1\!\!1^{+,-}$ and $1\!\!1^{-,+}$. The character $1\!\!1^{+,-}$ is characterized by

$$1\!\!1^{+,-}|_{O(k)} = \text{trivial}, \qquad 1\!\!1^{+,-}|_{O(l)} = \text{determinant}.$$

Likewise for $1\!\!1^{-,+}$.

We now highlight two of our results from §4.

Theorem 1.2. *Suppose that $p \geq q$, and consider the dual pairs*

$$(O(p, q), Sp(2q, \mathbb{R})), \quad \text{and} \quad (O(p - 1, q + 1), Sp(2q, \mathbb{R})).$$

We have

$$\theta^{p,q}_{\to q}(\mathbb{1}^{+,-}) \simeq \theta^{p-1,q+1}_{\to q}(\mathbb{1}).$$

The representations $\theta^{p,q}_{\to n}(\mathbb{1})$ (and in fact some larger representations containing $\theta^{p,q}_{\to n}(\mathbb{1})$ as unique irreducible quotients) are determined in [LZ2]. See §4.

Theorem 1.3. *Suppose that $p \geq q$, and consider the theta lift of $\mathbb{1}^{-,+}$ for the dual pair $(O(p,q), Sp(2p,\mathbb{R}))$. We have*

$$\theta^{p,q}_{\to p}(\mathbb{1}^{-,+}) \simeq \theta^{p+1,q-1}_{\to p}(\mathbb{1}) \simeq \theta^{p-q+2,0}_{\to p}(\mathbb{1}),$$

the unitary lowest weight representation of $\widetilde{Sp}(2p,\mathbb{R})$ with lowest weight $\frac{p-q+2}{2}\mathbf{1}_p$. Here $\mathbf{1}_p = \underbrace{(1,\cdots,1)}_{p}$.

Thus $\theta^{p,q}_{\to p}(\mathbb{1}^{-,+})$ is actually not a "strange fellow" (cf. [A2]).

Here is the organization of this paper. In §2, we review results of Shimura on invariant differential operators. In §3, we combine them with those of Harish-Chandra, Lepowsky-McCollum to arrive at Theorem 1.1. §4 is devoted to applications of Theorem 1.1 to local theta correspondence.

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2. Invariant differential operators (after Shimura)

Recall that G is a connected noncompact semisimple Lie group with finite center.

Given an irreducible finite dimensional representation $\rho : K \to GL(U)$, denote by $C^\infty(G;U)$ the space of U-valued C^∞ functions on G. Set

$$C^\infty(\rho) = \{f \in C^\infty(G;U) | f(gk^{-1}) = \rho(k)f(g), \ \forall g \in G, k \in K\}.$$

Denote by $\mathcal{D}(G)$ the ring of left-invariant differential operators on G. Then $\mathcal{D}(G)$ (respectively, $\mathcal{U}(\mathfrak{g})$) acts naturally on $C^\infty(G;U)$ through right translation (respectively, the derived action of right translation). Let

$$\mathcal{D}(\rho) = \{A \in \mathcal{D}(G) | A \cdot C^\infty(\rho) \subseteq C^\infty(\rho)\}, \qquad \text{and}$$
$$D(\rho) = \{B \in \mathcal{U}(\mathfrak{g}) | B \cdot C^\infty(\rho) \subseteq C^\infty(\rho)\}.$$

Clearly there is a natural surjective map

$$\pi : \quad D(\rho) \to \mathcal{D}(\rho).$$

Observe that for $g \in G$, $k \in K$, $B \in \mathfrak{g}$, $f \in C^\infty(\rho)$, we have

$$[(Ad(k)B)f](g) = \frac{d}{dt} f(g \cdot exp(tAd(k)B))|_{t=0}$$
$$= \frac{d}{dt} f(gk \cdot exp(tB)k^{-1})|_{t=0} = \rho(k)\frac{d}{dt} f(gk \cdot exp(tB))|_{t=0}$$
$$= \rho(k)(Bf)(gk).$$

The above equation remains true for $B \in \mathcal{U}(\mathfrak{g})$. In particular for $B \in \mathcal{U}(\mathfrak{g})^K$ (the K-invariants of $\mathcal{U}(\mathfrak{g})$), we have $(Bf)(g) = \rho(k)(Bf)(gk)$, namely $B \in D(\rho)$. We thus have

$$\mathcal{U}(\mathfrak{g})^K \subseteq D(\rho).$$

Observe that for $f \in C^\infty(\rho)$ and $X \in \mathfrak{k}_0$, we have $Xf = -d\rho(X)f$. We may extend $-d\rho$ uniquely to a $\mathbb{C}$-linear antihomomorphism of $\mathcal{U}(\mathfrak{k})$ into $\mathrm{End}(U)$, which will be denoted by $\rho_{\mathcal{U}}$. Then we have

$$Bf = \rho_{\mathcal{U}}(B)f, \qquad B \in \mathcal{U}(\mathfrak{k}),\ f \in C^\infty(\rho).$$

Let

$$\mathcal{R}_\rho = \ker(\rho_{\mathcal{U}}) \subseteq \mathcal{U}(\mathfrak{k})$$

be the kernel of $\rho_{\mathcal{U}}$.

Proposition 2.1. *(Shimura, [Sh])*

(a) *$B \in \mathcal{U}(\mathfrak{g})$ annihilates $C^\infty(\rho)$ if and only if $B \in \mathcal{U}(\mathfrak{g})\mathcal{R}_\rho$.*

(b) *$D(\rho) = \mathcal{U}(\mathfrak{g})^K + \mathcal{U}(\mathfrak{g})\mathcal{R}_\rho$.*

(c) *The natural map*

$$\pi : \quad D(\rho) \to \mathcal{D}(\rho)$$

induces an isomorphism of $\mathcal{U}(\mathfrak{g})^K / \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathcal{R}_\rho$ onto $\mathcal{D}(\rho)$.

Proposition 2.2. *(Shimura, [Sh]) If ρ is one dimensional, then*

(a) *$\mathcal{D}(\rho)$ is commutative.*

(b) *The bijection $\psi : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ induces a $\mathbb{C}$-linear bijection of $S(\mathfrak{p})^K$ onto $\mathcal{D}(\rho)$.*

(c) *$\psi(S^r(\mathfrak{p})^K)$ and $\mathcal{U}^r(\mathfrak{g})^K$ have the same image in $\mathcal{D}(\rho)$, for $r \in \mathbb{Z}_{\geq 0}$.*

Remark 2.3. (a) $\mathcal{U}(\mathfrak{g})^K / \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathcal{R}_\rho$ is in fact anti-isomorphic to a subalgebra of $\mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k})/\mathcal{R}_\rho$, where $\mathfrak{a} \subseteq \mathfrak{p}$ is an abelian subalgebra of $\mathfrak{g}$. See [Le, Theorem 1.3].

(b) Suppose that ρ is finite dimensional and irreducible. For $\sum_i E_i \otimes B_i \in \mathrm{End}(U) \otimes S(\mathfrak{p})$, define $\Psi(\sum_i E_i \otimes B_i)$ to be the operator on $C^\infty(G, U)$ given by

$$\Psi(\sum_i E_i \otimes B_i) = \sum_i E_i \psi(B_i).$$

Then Ψ induces a bijection of $(\mathrm{End}(U) \otimes S(\mathfrak{p}))^K$ onto $\mathcal{D}(\rho)$. See [Sh, Proposition 2.2].

Let $Z(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$. Clearly

$$Z(\mathfrak{g}) \subseteq \mathcal{U}(\mathfrak{g})^K \subseteq D(\rho).$$

Theorem 2.4. *(Shimura, [Sh]) Suppose that*

$$\psi(S^r(\mathfrak{g})^G) + \mathcal{U}(\mathfrak{g})\mathfrak{k} \supseteq \psi(S^r(\mathfrak{p})^K), \qquad r \in \mathbb{Z}_{\geq 0} \tag{2.5}$$

(true at least for all classical G), then the natural map

$$\pi|_{Z(\mathfrak{g})} : \quad Z(\mathfrak{g}) \to \mathcal{D}(\rho)$$

is surjective for ρ one-dimensional.

Remark 2.6. For $\rho = \mathbb{1}$, the trivial representation, the we have $\mathcal{D}(\rho) = \mathcal{D}(G/K)$, the algebra of G-invariant differential operators on the symmetric space G/K. Here are some historical remarks on the surjectivity of

$$Z(\mathfrak{g}) \to \mathcal{D}(G/K).$$

(a) Berezin stated (incorrectly) the surjectivity (1957).

(b) The mistake was pointed out by Helgason (1964), and he proved that the surjectivity is true for classical groups, but not true for some real forms of $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ [He1].

(c) Shimura later pointed out a gap in Helgason's proof. A correct proof was given by Helgason [He2, 1989] who showed that $Z(\mathfrak{g}) \to \mathcal{D}(G/K)$ is surjective if and only if G does not contain any simple factor of the following four exceptional types: $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{6(-24)}$, $\mathfrak{e}_{7(-25)}$, $\mathfrak{e}_{8(-24)}$.

Remark 2.7. If G is simple, then K has a non-trivial one-dimensional representation if and only if the pair (G, K) is of Hermitian symmetric type. Suppose this is the case and suppose that G is exceptional, namely G is of type $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(-25)}$, then the condition (2.5) will not hold. This is because its validity will imply in particular the surjectivity of $Z(\mathfrak{g}) \to \mathcal{D}(G/K)$, which is false by the result of Helgason. Thus the additional cases covered by Theorem 2.4 (apart from $\rho = \mathbb{1}$) are really for G classical and of Hermitian symmetric type.

3. Admissible representations with scalar K-types

Let (π, V) be an irreducible admissible representation of G. Denote by V^∞ the space of smooth vectors of V, and V^∞_ρ the ρ-isotypic component of V^∞, where $\rho \in \hat{K}$. The Harish-Chandra module of (π, V) is then the algebraic direct sum

$$V_K = \sum_{\rho \in \hat{K}} V_\rho^\infty.$$

For each $\rho \in \hat{K}$, the space V_ρ^∞ is naturally a $\mathcal{U}(\mathfrak{g})^K \times K$ module.

Theorem 3.1. (a) *(Harish-Chandra, [Ha]) If $V_\rho^\infty \neq 0$, then the $\mathcal{U}(\mathfrak{g})^K$ module structure of V_ρ^∞ determines π up to infinitesimal equivalence.*

(b) *(Lepowsky-McCollum, [LM]) Denote $I_\rho = \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathcal{R}_\rho$. Then I_ρ is a two-sided ideal in $\mathcal{U}(\mathfrak{g})^K$, and the action of $\mathcal{U}(\mathfrak{g})^K$ on V_ρ^∞ factors through $\mathcal{U}(\mathfrak{g})^K/I_\rho$. This establishes a one-to-one correspondence between irreducible admissible representations V of G with $V_\rho^\infty \neq 0$ and irreducible $\mathcal{U}(\mathfrak{g})^K/I_\rho$ modules.*

We are now ready to prove the following

Theorem 3.2. *Suppose that*

$$\psi(S^r(\mathfrak{g})^G) + \mathcal{U}(\mathfrak{g})\mathfrak{k} \supseteq \psi(S^r(\mathfrak{p})^K), \qquad r \in \mathbb{Z}_{\geq 0}$$

(true at least for all classical G), and $\rho \in \hat{K}$ is one-dimensional. For an irreducible Harish-Chandra module V with $V_\rho \neq 0$, the infinitesimal character of V determines V up to infinitesimal equivalence.

Proof. By Proposition 2.1, we have

$$\mathcal{D}(\rho) \cong \mathcal{U}(\mathfrak{g})^K/I_\rho.$$

From Theorem 2.4, we know that the natural map

$$Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^K/I_\rho$$

is surjective under the current hypothesis. Therefore for $B \in \mathcal{U}(\mathfrak{g})^K$, there exists $Z_B \in Z(\mathfrak{g})$ such that

$$B - Z_B \in I_\rho.$$

Clearly given $B \in \mathcal{U}(\mathfrak{g})^K$, such Z_B is unique modulo $Z(\mathfrak{g}) \cap I_\rho$.

Let χ be the infinitesimal character of V. Then we have

$$B \cdot v = \chi(Z_B)v, \qquad B \in \mathcal{U}(\mathfrak{g})^K, v \in V_\rho.$$

By the Theorem of Harish-Chandra (Theorem 3.1 (a)), we conclude that the infinitesimal character χ of V determines V up to infinitesimal equivalence. □

4. Applications to local theta correspondence

Fix a reductive dual pair [H1]

$$(G', G) \subseteq Sp(2N, \mathbb{R}).$$

Let $\widetilde{Sp}(2N, \mathbb{R})$ be the metaplectic cover of $Sp(2N, \mathbb{R})$, and for any subgroup L of $Sp(2N, \mathbb{R})$, let $\widetilde{L}$ be the inverse image of L in $\widetilde{Sp}(2N, \mathbb{R})$. Denote by K' (resp. K) a maximal compact subgroup of G' (resp. G).

We also fix a (smooth) oscillator representation Ω of $\widetilde{Sp}(2N, \mathbb{R})$. Recall [H2] that an irreducible admissible representation π' of $\widetilde{G'}$ and an irreducible admissible representation π of $\widetilde{G}$ are said to correspond to each other under the local theta correspondence (or the Howe duality correspondence) if there exists a non-zero $\widetilde{G'} \times \widetilde{G}$-intertwining map

$$\Omega \mapsto \pi' \otimes \pi.$$

In that case we write

$$\theta(\pi') = \pi, \qquad \text{or} \qquad \theta(\pi) = \pi',$$

or in a symmetric way $\pi' \leftrightarrow \pi$.

A basic property of the local theta correspondence is that it has subordinated to it a correspondence of certain $\widetilde{K'}$ and $\widetilde{K}$ types, which we shall describe.

Recall that the Fock model of Ω is realized in the space $\mathcal{F}$ of polynomials in N variables. Using this, we may associate to each $\widetilde{K'}$- and $\widetilde{K}$-type occuring in $\mathcal{F}$ a degree, which will be called the Howe degree. Let $\mathcal{H}$ be the space of joint $\widetilde{K'}$ and $\widetilde{K}$ harmonics ([H2]). This is a $\widetilde{K'} \times \widetilde{K}$-invariant subspace of $\mathcal{F}$.

Theorem 4.1. *(Howe, [H2]) There is a one to one correspondence of $\widetilde{K'}$- and $\widetilde{K}$-types in $\mathcal{H}$ with the following properties. Suppose that π' and π are irreducible admissible representations of $\widetilde{G'}$ and $\widetilde{G}$ respectively, and $\pi' \leftrightarrow \pi$ in the correspondence for the dual pair (G', G). Let σ' be a $\widetilde{K'}$-type occuring in π', and suppose that σ' is of minimal Howe degree among all the $\widetilde{K'}$-types of π'. Let σ be the $\widetilde{K}$-type which corresponds to σ' in $\mathcal{H}$. Then σ is a $\widetilde{K}$-type of minimal Howe degree in π.*

In what follows, we shall only examine the dual pair

$$(O(p, q), Sp(2n, \mathbb{R})) \subseteq Sp(2N, \mathbb{R}),$$

where $N = 2(p + q)n$. The theta lift from $O(p, q)$ to $Sp(2n, \mathbb{R})$ will be denoted by $\theta^{p,q}_{\to n}$, as in the Introduction.

We have $K' = O(p) \times O(q)$ and $K = U(n)$. The following description of correspondence of K'- and $\widetilde{K}$-types in $\mathcal{H}$ is well-known. See for example [A1]. We shall write this correspondence as $\sigma' \leftrightarrow \sigma$. Recall that irreducible representations of $O(p)$ are indexed by $(a_1, ..., a_k, 0, ..., 0; \epsilon)$, where $(a_1, ..., a_k, 0, ..., 0)$ are the usual highest weights of irreducible representations of $SO(p)$, and $\epsilon = \pm 1$. For $\widetilde{U(n)}$, irreducible representations are indexed by their highest weights, with their components integers or half integers. Denote as before

$$\mathbf{1}_n = \underbrace{(1, \cdots, 1)}_{n}.$$

Fact 4.2. *Correspondence of joint harmonics for* $(O(p,q), Sp(2n, \mathbb{R}))$

$$\sigma' = (a_1, ..., a_k, 0, ..., 0; \epsilon) \otimes (b_1, ..., b_l, 0, ..., 0; \eta) \leftrightarrow$$

$$\sigma = \frac{p-q}{2}\mathbf{1}_n + (a_1, ..., a_k, \underbrace{1, ..., 1}_{\frac{1-\epsilon}{2}(p-2k)}, 0, ..., 0, \underbrace{-1, ..., -1}_{\frac{1-\eta}{2}(q-2l)}, -b_l, ..., -b_1),$$

where $2k \le p$, $2l \le q$ *and* $k + \frac{1-\epsilon}{2}(p-2k) + l + \frac{1-\eta}{2}(q-2l) \le n$.

Denote $m = p + q$. Recall also the correspondence of infinitesimal characters under the local theta correspondence [Pr]. In particular we have

Fact 4.3. *For any character* χ *of* $O(p,q)$, *the theta lift* $\theta^{p,q}_{\to n}(\chi)$ *has the infinitesimal character with Harish-Chandra parameter* $(\frac{m}{2}-1, \frac{m}{2}-2, \cdots, \frac{m}{2}-n)$.

4.1. **The trivial character $\mathbb{1}$.** The trivial character $\mathbb{1}$ occurs in the local theta correspondence of $(O(p,q), Sp(2n, \mathbb{R}))$ for any n. See [Zh] for example.

Note that $\theta^{m,0}_{\to n}(\mathbb{1})$ (resp. $\theta^{0,m}_{\to n}(\mathbb{1})$) is the unitary lowest (resp. highest) weight representation of $\widetilde{Sp}(2n, \mathbb{R})$ with lowest weight $\frac{m}{2}\mathbf{1}_n$ (resp. highest weight $-\frac{m}{2}\mathbf{1}_n$).

The representations $\theta^{p,q}_{\to n}(\mathbb{1})$ are very well understood. In fact $\theta^{p,q}_{\to n}(\mathbb{1})$ is the unique irreducible quotient of a certain natural $\widetilde{Sp}(2n, \mathbb{R})$-module $\Omega^{p,q}_{\to n}(\mathbb{1})$, called the Howe (maximal) quotient corresponding to the trivial representation of $O(p,q)$, and complete structures of $\Omega^{p,q}_{\to n}(\mathbb{1})$ (and therefore $\theta^{p,q}_{\to n}(\mathbb{1})$) are known [LZ2]. In particular, $\Omega^{p,q}_{\to n}(\mathbb{1})$ is $\widetilde{K}$ multiplicity-free, and all $\widetilde{K}$-types of $\Omega^{p,q}_{\to n}(\mathbb{1})$ and of their irreducible constituents are explicitly described. One may also write down their Langlands parameters (see [PT] for the unitary case).

The following proposition summaries some basic and relevant properties of $\theta^{p,q}_{\to n}(\mathbb{1})$. For $\sigma \in \mathbb{C}$ and $\alpha = 0, 1, 2, 3$, let $I(\sigma; \alpha)$ be the degenerate principal series of $\widetilde{Sp}(2n,\mathbb{R})$ (normalized) induced from the character $\chi_0^\alpha \otimes |\cdot|^\sigma$ of the Siegel parabolic subgroup. Here χ_0 is a certain character of order 4 (roughly the square root of the sign of determinant).

Theorem 4.1.1. *(see [LZ2])*
(a) $\theta^{p,q}_{\to n}(\mathbb{1})$ *is isomorphic to the unique irreducible constituent of* $I(\sigma;\alpha)$ *containing the scalar* $\widetilde{K}$*-type* $\frac{p-q}{2}\mathbf{1}_n$*, where* $\sigma = \frac{m-(n+1)}{2}$ *and* $\alpha \equiv p-q$ *(mod 4).*
(b) $\theta^{p,q}_{\to n}(\mathbb{1})$ *is finite dimensional if and only if* $p,q \equiv n+1$ *(mod 2), and* $p,q \geq n+1$*. In this case,* $\theta^{p,q}_{\to n}(\mathbb{1})$ *is isomorphic to* $F_{(\frac{m}{2}-n-1)\mathbf{1}_n}$*, the finite dimensional irreducible representation with the highest weight* $(\frac{m}{2}-n-1)\mathbf{1}_n$.
(c) $\theta^{p,q}_{\to n}(\mathbb{1})$ *is unitary if and only if either* $pq=0$ *or both* $p,q \leq n+1$.

The following proposition may be read off from the results of [LZ2]. We give a proof in the spirit of this article.

Proposition 4.1.2. *Let* $p,q \leq n+1$*, then we have*

$$\theta^{n+1-q,n+1-p}_{\to n}(\mathbb{1}) \simeq \theta^{p,q}_{\to n}(\mathbb{1}).$$

Proof. Both $\theta^{n+1-q,n+1-p}_{\to n}(\mathbb{1})$ and $\theta^{p,q}_{\to n}(\mathbb{1})$ contain the scalar $\widetilde{K}$-type $\frac{p-q}{2}\mathbf{1}_n$ (Fact 4.2). The infinitesimal character of the latter has Harish-Chandra parameter $(\frac{m}{2}-1, \frac{m}{2}-2, \cdots, \frac{m}{2}-n)$, while that of the former has Harish-Chandra parameter $(n-\frac{m}{2}, ..., 1-\frac{m}{2})$. Since they are conjugate via a Weyl group element, the assertion clearly follows from Theorem 3.2. ☐

4.2. **The characters $\mathbb{1}^{+,-}$ and $\mathbb{1}^{-,+}$.** Without any loss of generality, we may assume that $p \geq q$. From the correspondence of joint harmonics (Fact 4.2), we see that $\mathbb{1}^{+,-}$ occurs in the local theta correspondence for the dual pair $(O(p,q), Sp(2n,\mathbb{R}))$ only if $q \leq n$. At $n = q$, we know from the existance of certain tempered $\mathbb{1}^{+,-}$-eigendistribution [HZ] that $\mathbb{1}^{+,-}$ does occur. In view of the persistence principle of Kudla [K], we may conclude that

Proposition 4.2.1. *The character* $\mathbb{1}^{+,-}$ *occurs in the local theta correspondence for the dual pair* $(O(p,q), Sp(2n,\mathbb{R}))$ *if and only if* $q \leq n$.

At $n = q$, it will be refered to as the point of first occurrence for $\mathbb{1}^{+,-}$. From Fact 4.2, we see that $\theta^{p,q}_{\to q}(\mathbb{1}^{+,-})$ contains the scalar $\widetilde{K}$-type $\frac{p-q-2}{2}\mathbf{1}_q$. From Fact 4.3 and Theorem 3.2, we immediately have

Theorem 4.2.2. *Suppose that* $p \geq q$*, and consider the dual pairs*

$$(O(p,q), Sp(2q,\mathbb{R})), \quad \textit{and} \quad (O(p-1,q+1), Sp(2q,\mathbb{R})).$$

We have

$$\theta^{p,q}_{\to q}(1\!\!1^{+,-}) \simeq \theta^{p-1,q+1}_{\to q}(1\!\!1).$$

We may then combine Theorem 4.1.1 with Theoem 4.2.2 to obtain the following

Corollary 4.2.3. (a) $\theta^{p,q}_{\to q}(1\!\!1^{+,-})$ *is finite dimensional if and only if* $p \equiv q$ *(mod 2), and* $p > q$*. In this case,* $\theta^{p,q}_{\to q}(1\!\!1^{+,-})$ *is isomorphic to* $F_{(\frac{p-q}{2}-1)\mathbf{1}_q}$.

(b) $\theta^{p,q}_{\to q}(1\!\!1^{+,-})$ *is unitary if and only if* $p = q, q+1, q+2$*, and*

$$\theta^{p,q}_{\to q}(1\!\!1^{+,-}) \simeq \begin{cases} \theta^{0,2}_{\to q}(1\!\!1), & p = q, \\ \theta^{0,1}_{\to q}(1\!\!1), & p = q+1, \\ 1\!\!1, & p = q+2. \end{cases}$$

Remark 4.2.4. The assertion in Corollary 4.2.3 (a) is related to certain interesting phenomenon about solutions of a system of generalized wave equations, called the "Generalized Huygens' Principle". See [HZ].

Now we examine the character $1\!\!1^{-,+}$. Similarly for $1\!\!1^{-,+}$ to occur in the local theta correspondence for the dual pair $(O(p,q), Sp(2n,\mathbb{R}))$, we must have $p \leq n$. The next proposition implies that it does occur at $n = p$.

Proposition 4.2.5. *There exists a non-zero tempered* $1\!\!1^{-,+}$*-eigendistribution of* $O(p,q)$ *on the matrix space* $M_{p+q,p}(\mathbb{R})$*. Hence* $1\!\!1^{+,-}$ *occurs for the dual pair* $(O(p,q), Sp(2n,\mathbb{R}))$ *if and only if* $p \leq n$*.*

Proof. We consider a larger matrix space $M_{2p+2,p}(\mathbb{R})$. According to [HZ], there exists a non-zero tempered $1\!\!1^{-,+}$-eigendistribution Ψ of $O(p,p+2)$ on $M_{2p+2,p}(\mathbb{R})$. In the notation of [HZ], $\Psi = d\nu_p$ is the difference of two $O^+(p,p+2)$ invariant measures supported on the $O(p,p+2)$ nullcone in $M_{2p+2,p}(\mathbb{R})$, where $O^+(p,p+2)$ is a subgroup of $O(p,p+2)$ of index 2. For the case under consideration, the distribution Ψ is invariant under $\widetilde{Sp}(2p,\mathbb{R})$ (through the smooth oscillator representation Ω of $\widetilde{Sp}(2p(2p+2),\mathbb{R})$ on the Schwarz space $\mathcal{S}(M_{2p+2,p}(\mathbb{R}))$). C.f. Corollary 4.2.3 (b). But we shall not use this fact.

Write a typical element $v \in M_{2p+2,p}(\mathbb{R})$ as $v = \begin{pmatrix} a \\ b \end{pmatrix}$, where

$$a = (a_{ij})_{(p+q)\times p} \in M_{p+q,p}(\mathbb{R}), \text{ and } b = (b_{ij})_{(p-q+2)\times p} \in M_{p-q+2,p}(\mathbb{R}).$$

Corresponding to this, we have the tensor product decomposition

$$\mathcal{S}(M_{2p+2,p}(\mathbb{R})) \simeq \mathcal{S}(M_{p+q,p}(\mathbb{R})) \otimes \mathcal{S}(M_{p-q+2,p}(\mathbb{R})).$$

Since $\Psi \neq 0$, we see that there exists a $\phi \in \mathcal{S}(M_{p-q+2,p}(\mathbb{R}))$ such that

$$\Psi|_{\mathcal{S}(M_{p+q,p}(\mathbb{R}))\otimes\phi} \neq 0.$$

This restriction will then define a non-zero tempered distribution on $M_{p+q,p}(\mathbb{R})$, which is clearly a $1\!\!1^{-,+}$-eigendistribution of $O(p,q)$. □

Remark 4.2.6. It is not difficult to see that we may take

$$\phi(b) = \exp(-tr(b^t b)/2), \qquad b \in M_{p-q+2,p}(\mathbb{R})$$

in the above proof.

Theorem 4.2.7. *Suppose that $p \geq q$, and consider the theta lift of $1\!\!1^{-,+}$ for the dual pair $(O(p,q), Sp(2p,\mathbb{R}))$. We have*

$$\theta^{p,q}_{\to p}(1\!\!1^{-,+}) \simeq \theta^{p+1,q-1}_{\to p}(1\!\!1) \simeq \theta^{p-q+2,0}_{\to p}(1\!\!1),$$

the unitary lowest weight representation of $\widetilde{Sp}(2p,\mathbb{R})$ with lowest weight $\frac{p-q+2}{2}\mathbf{1}_p$.

Proof. From the correspondence of joint harmonics (Fact 4.2), we see that $\theta^{p,q}_{\to p}(1\!\!1^{-,+})$ contains the scalar $\widetilde{K}$-type $\frac{p-q+2}{2}\mathbf{1}_p$. Since $\theta^{p+1,q-1}_{\to p}(1\!\!1)$ contains the same scalar $\widetilde{K}$-type, and since the two representations clearly have the same infinitesimal character, the first isomorphism follows from Theorem 3.2. The second isomorphism is a special case of Proposition 4.1.2. □

4.3. **Remarks about the determinant character.** The determinant character det of $O(p,q)$ occurs in the local theta correspondence for the dual pair $(O(p,q), Sp(2n,\mathbb{R}))$ if and only if $p+q \leq n$. As in the case of $\theta^{p,q}_{\to n}(1\!\!1)$, the representation $\theta^{p,q}_{\to n}(\det)$ is $\widetilde{K}$ multiplicity-free, and all its $\widetilde{K}$-types can be explicitly described. See [LZ2].

The point of first occurance for det is thus at $n = p+q$, which is in the so-called stable range. Assume this is the case. From Fact 4.2, we know that $\theta^{p,q}_{\to n}(\det)$ contains the $\widetilde{K}$-type

$$\frac{p-q}{2}\mathbf{1}_n + (\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q}).$$

It turns out that $\theta^{p,q}_{\to n}(\det)$ contains a scalar $\widetilde{K}$-type if and only if $pq = 0$. When this happens, clearly we have

$$\theta^{n,0}_{\to n}(\det) \simeq \theta^{n+2,0}_{\to n}(1\!\!1), \qquad \theta^{0,n}_{\to n}(\det) \simeq \theta^{0,n+2}_{\to n}(1\!\!1).$$

In all other cases, namely $p+q = n$ and $p,q > 0$, the representation $\theta^{p,q}_{\to n}(\det)$ is isomorphic to certain submodule of $\Omega^{p',q'}_{\to n}(1\!\!1)$, where $p'+q' = n+2$. See [LZ2], Theorems 4.15 and 4.17. Note that $\theta^{p',q'}_{\to n}(1\!\!1)$ is the unique irreducible quotient of $\Omega^{p',q'}_{\to n}(1\!\!1)$.

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