Multiplicity-free actions and the geometry of nilpotent orbits

Kyo Nishiyama

Faculty of IHS, Kyoto Univeristy
Sakyo, Kyoto 606-8501, Japan
e-mail: kyo@math.h.kyoto-u.ac.jp

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Introduction

Let G be a reductive Lie group. Take a maximal compact subgroup K of G and denote their Lie algebras by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. We get a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ the complexified decomposition. The adjoint action restricted to K preserves the space $\mathfrak{s}_0$, hence $K_{\mathbb{C}}$ acts on $\mathfrak{s}$, where $K_{\mathbb{C}}$ denotes a complexification of K.

In this paper, we consider a series of small nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{s}$ which are obtained from the dual pair $(G, G') = (O(p,q), Sp(2n, \mathbb{R}))$ (cf. [4]).

We explain astonishing simple structures of these nilpotent orbits using generalized null cones (see §2 for the definition). For example, these orbits have a linear ordering with respect to the closure relation (Theorem 4.1); and $K_{\mathbb{C}}$ acts on them in multiplicity-free manner. We clarify the $K_{\mathbb{C}}$-module structure of the regular function ring of the closure of these nilpotent orbits in detail (Corollary 5.3), and prove the normality (Corollary 5.2). All these results naturally comes from the analysis on the null cone $\mathfrak{N}$ (Theorems 2.1, 3.1 and 3.3) and the classical invariant theory. We also provide many interesting examples of multiplicity-free actions on conic algebraic varieties.

Let $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ be a complexified Cartan decomposition of $\mathfrak{g}' = \mathrm{Lie}(G')_{\mathbb{C}}$. The key idea of the construction of the nilpotent orbits relies on the so-called *unfolding maps* φ_1 and φ_2 (see §1). We construct a double fibration of $\mathfrak{s} \subset \mathfrak{g}$ and $\mathfrak{s}' \subset \mathfrak{g}'$ by using φ_1 and φ_2. In this way, we can consider the closure of a nilpotent $K_{\mathbb{C}}$-orbit $\overline{\mathcal{O}_n}$ as a geometric quotient of the null cone $\mathfrak{N}$ by $K'_{\mathbb{C}}$: $\overline{\mathcal{O}_n} \simeq \mathfrak{N}//K'_{\mathbb{C}}$. On the other hand, the null cone itself is defined to be the inverse image of the origin of $\mathfrak{s}'$ via the map φ_2, so we can regard the nilpotent orbit $\mathcal{O}_n \subset \mathfrak{s}$ as the "lift" of the trivial orbit in $\mathfrak{s}'$. In fact, we have $\overline{\mathcal{O}_n} = \varphi_1(\mathfrak{N}) = \varphi_1(\varphi_2^{-1}(0))$.

We also consider small nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{s}'$ and get similar results.

Let π be an irreducible admissible representation of G on a Hilbert space. Then the associated cycle of π is a finite union of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{s}$ together with multiplicities of the maximal orbits (see [8]). The geometry of the associated cycle (or, more simply, the associated variety) reflects various properties of the representation π. For example, the dimension of the associated variety is equal to the Gelfand-Kirillov dimension of π. Moreover, the Bernstein degree of π can be computed from the associated cycle ([6]).

We will use the results in this article to study the theta lift of the trivial representations for the dual pair $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$ ([9]; see also [4]). The associated cycle of the theta lift of the trivial representation is the closure of an orbit obtained here without multiplicity. The details will appear in a forthcoming paper.

Let us explain each section briefly. Let $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$ be a dual pair. We will denote by K' a maximal compact subgroup of G' and by $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ a complexified Cartan decomposition.

In §1, we define the two unfolding maps φ_1 and φ_2. These maps relates the nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{s}$ and $\mathfrak{s}'$ tightly. In §§2 and 3, we study the geometry of the null cone $\mathfrak{N} = \varphi_2^{-1}(0)$. In these sections, we assume that $2n < \min(p, q)$, which means that the pair (G, G') is in the stable range with G' being the smaller member. We prove the null cone $\mathfrak{N}$ is an irreducible affine cone of complete intersection. Moreover, we construct an algebraic group which acts on $\mathfrak{N}$ in a multiplicity-free manner. In §§4 and 5, we consider the closure of a nilpotent orbit $\overline{\mathcal{O}_n}$ which is the image of the null cone via the unfolding map φ_1: $\overline{\mathcal{O}_n} = \varphi_1(\mathfrak{N})$. The orbit $\mathcal{O}_n$ is regarded as the theta lift of the trivial orbit in $\mathfrak{s}'$ for $G' = Sp(2n, \mathbb{R})$. We prove its normality under the condition $2n + 1 < \min(p, q)$, and $K_{\mathbb{C}}$-module structure of its regular function ring.

Finally, in §6, we assume that $p + q < n$ and consider the nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}_{p,q}$ in $\mathfrak{s}'$. This orbit is the theta lift of the trivial orbit in $\mathfrak{s}$ for $G = O(p, q)$. Therefore, we consider the "reversed" situation in a sense. In this section, we only state the results since the method is almost the same as in the former sections.

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1 Nilpotent orbits

Let us consider two Riemannian symmetric pairs

$$G/K = O(p,q)/O(p) \times O(q) \quad \text{and} \quad G'/K' = Sp(2n,\mathbb{R})/U(n).$$

We denote $\mathfrak{k} = \mathfrak{o}(p)_{\mathbb{C}} \oplus \mathfrak{o}(q)_{\mathbb{C}}$ and $\mathfrak{k}' = \mathfrak{gl}(n,\mathbb{C})$, the complexified Lie algebra of K and K' respectively. Let $\mathfrak{g} = \mathfrak{o}(p+q,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{s}$ and $\mathfrak{g}' = \mathfrak{sp}(2n,\mathbb{C}) = \mathfrak{k}' \oplus \mathfrak{s}'$ be Cartan decompositions. Namely, we realize

$$\mathfrak{g} = \mathfrak{o}(p+q,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ {}^tB & D \end{bmatrix} \mid A \in \operatorname{Alt}_p, D \in \operatorname{Alt}_q, B \in M_{p,q} \right\},$$

$$\mathfrak{g}' = \mathfrak{sp}(2n,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & -{}^tA \end{bmatrix} \mid A \in M_n, B, C \in \operatorname{Sym}_n \right\},$$

where $M_{p,q} = M_{p,q}(\mathbb{C})$ denotes the space of all $n \times m$-matrices, and $\operatorname{Sym}_n = \operatorname{Sym}_n(\mathbb{C})$ (respectively $\operatorname{Alt}_n = \operatorname{Alt}_n(\mathbb{C})$) denotes the space of all symmetric (respectively alternating) matrices of size n. Note that $\mathfrak{sp}(2n,\mathbb{C})$ here is *not* realized in the standard way (see [6, § 7.1], for example). Thus we can identify $\mathfrak{s} = M_{p,q}$ and $\mathfrak{s}' = \operatorname{Sym}_n \oplus \operatorname{Sym}_n$.

Let us consider the following three spaces:

$$W = M_{p,n} \oplus M_{q,n}, \quad \mathfrak{s} = M_{p,q}, \quad \text{and} \quad \mathfrak{s}' = \operatorname{Sym}_n \oplus \operatorname{Sym}_n.$$

We define two maps among these spaces as follows. Take $(A,B) \in M_{p,n} \oplus M_{q,n} = W$. Then we put

$$\begin{cases} \varphi_1(A,B) = A\,{}^tB \in M_{p,q} = \mathfrak{s} \quad \text{and} \\ \varphi_2(A,B) = ({}^tAA, {}^tBB) \in \operatorname{Sym}_n \oplus \operatorname{Sym}_n = \mathfrak{s}'. \end{cases}$$

It can be readily checked that these maps are well-defined and homogeneous of degree two.

$$\begin{array}{ccc} & W = M_{p,n} \oplus M_{q,n} & \\ & {}^{\varphi_1}\swarrow \quad \searrow^{\varphi_2} & \\ \mathfrak{s} = M_{p,q} & & \operatorname{Sym}_n \oplus \operatorname{Sym}_n = \mathfrak{s}' \end{array}$$

Let us consider the following four algebraic groups over the field of complex numbers $\mathbb{C}$,

$$\begin{array}{ccc} GL_p \times GL_q & =: & M_{\mathbb{C}} \\ \cup & & \\ O_p \times O_q & =: & K_{\mathbb{C}} \end{array} \quad \text{and} \quad \begin{array}{ccc} GL_n \times GL_n & =: & M'_{\mathbb{C}} \\ \cup & & \\ \Delta GL_n & =: & K'_{\mathbb{C}} \end{array}$$

so that $K_{\mathbb{C}}$ (respectively $K'_{\mathbb{C}}$) is a spherical subgroup of $M_{\mathbb{C}}$ (respectively $M'_{\mathbb{C}}$) (in fact, these are symmetric pairs). We identify $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$ with the complexification of K and K' respectively.

We define linear actions of these groups on $W, \mathfrak{s}$ and $\mathfrak{s}'$ as usual. To be more precise, let us write down explicit actions. Take an $(A, B) \in W$. Then an element $g = ((g_1, h_1), (g_2, h_2)) \in M_{\mathbb{C}} \times M'_{\mathbb{C}}$ acts on $(A, B) \in M_{p,n} \oplus M_{q,n}$ as

$$g \cdot (A, B) = (g_1 A\, {}^t g_2, h_1 B h_2^{-1}).$$

For $\mathfrak{s}$, $(g_1, h_1) \in M_{\mathbb{C}}$ acts on $X \in M_{p,q}$ as

$$(g_1, h_1) \cdot X = g_1 X\, {}^t h_1,$$

while $M'_{\mathbb{C}}$ acts on $\mathfrak{s}$ trivially. For $\mathfrak{s}'$, $(g_2, h_2) \in M'_{\mathbb{C}}$ acts on $(C, D) \in \mathrm{Sym}_n \oplus \mathrm{Sym}_n$ as

$$(g_2, h_2) \cdot (C, D) = (g_2 C\, {}^t g_2,\ {}^t h_2{}^{-1} C h_2^{-1}),$$

while $M_{\mathbb{C}}$ acts on $\mathfrak{s}'$ trivially. As a subgroup, $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$ also act linearly on these spaces. It is easy to check that the map φ_1 is $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant, and φ_2 is $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-equivariant. Note that the action of $K_{\mathbb{C}}$ on $\mathfrak{s}$ coincides with the adjoint action of $K_{\mathbb{C}}$. The same remark is applicable to the action of $K'_{\mathbb{C}}$.

The maps φ_1 and φ_2 are closely related to the nilpotent $\mathrm{Ad}\, K_{\mathbb{C}}$-orbits in $\mathfrak{s}$.

Proposition 1.1 *Let $\mathcal{N}_{\mathfrak{s}}$ (respectively $\mathcal{N}_{\mathfrak{s}'}$) be the set of all nilpotent elements in $\mathfrak{s}$ (respectively $\mathfrak{s}'$). Then we have*

$$\varphi_1(\varphi_2^{-1}(\mathcal{N}_{\mathfrak{s}'})) \subset \mathcal{N}_{\mathfrak{s}} \quad \textit{and} \quad \varphi_2(\varphi_1^{-1}(\mathcal{N}_{\mathfrak{s}})) \subset \mathcal{N}_{\mathfrak{s}'}.$$

PROOF. First we note the following easy observation.

Lemma 1.2 *Let us consider an element*

$$Y = \begin{bmatrix} 0 & B \\ {}^t B & 0 \end{bmatrix} \in \mathfrak{s}.$$

Then Y is nilpotent if and only if one of $B\, {}^t B$ or ${}^t B B$ is nilpotent. Similarly, let us consider an element

$$X = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathfrak{s}'.$$

Then X is nilpotent if and only if one of BC or CB is nilpotent.

Take $(A, B) \in \varphi_2^{-1}(X) \subset W$. Then ${}^t A A\, {}^t B B$ and ${}^t B B\, {}^t A A$ are both nilpotent. Now, since $\varphi_2(A, B) = A\, {}^t B$, it is clear that

$$\varphi_2(A, B)\, {}^t \varphi_2(A, B) = A\, {}^t B B\, {}^t A$$

is nilpotent. This implies $\varphi_2(A, B)$ is nilpotent.

The proof for Y is similar. Q.E.D.

2 Null cone

Now *we assume that* $2n < \min(p,q)$. This assumption is essential in the following arguments.

Put $\mathfrak{N} = \varphi_2^{-1}(0)$ and call it a *null cone*. Since φ_2 is a $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-equivariant map, $\mathfrak{N}$ is clearly $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-stable. Moreover, we have

Theorem 2.1 *We assume* $\min(p,q) > 2n$.

(1) *The null cone* $\mathfrak{N}$ *is the closure of a single* $K_{\mathbb{C}}^{\circ} \times M'_{\mathbb{C}}$*-orbit in* W*, where* $K_{\mathbb{C}}^{\circ}$ *denotes the connected component of the identity element. In particular,* $\mathfrak{N}$ *is an irreducible affine variety.*

(2) $\mathfrak{N}$ *is a variety of complete intersection, and* $\dim \mathfrak{N} = n(p+q) - n(n+1)$*. In particular, the regular function ring* $\mathbb{C}[\mathfrak{N}]$ *is Cohen-Macaulay.*

(3) *If* $\min(p,q) \geq 2 + 2n$ *holds,* $\mathfrak{N}$ *is a normal variety.*

In the rest of this section, we shall prove this theorem.

First, we make a few remarks. Put $W^+ = M_{p,n}$ and $W^- = M_{q,n}$. Then $W = W^+ \oplus W^-$ is a direct sum. We denote the restriction of φ_2 on $W^{\pm}$ by $\varphi_2^{\pm}$:

$$\begin{array}{cccc} \varphi_2^{\pm} : & W^{\pm} & \longrightarrow & \mathrm{Sym}_n \\ & \Cup & & \Cup \\ & Z & \longmapsto & {}^tZZ \end{array} .$$

Then $\mathfrak{N}^{\pm} = (\varphi_2^{\pm})^{-1}(0)$ is also called a null cone, and abviously $\mathfrak{N}$ is a direct product of these two null cones $\mathfrak{N}^{\pm}$. Therefore, our theorem above will follow from the results for $\mathfrak{N}^{\pm}$ (for example, see [5, § 3.7.3] and the references there). However, we do not prove the theorem in such a way. Our proof here is direct, and it will clarify the structure of $\mathfrak{N}$ considerably.

Put

$$\mathfrak{N}_{k,l} = \{(A,B) \in \mathfrak{N} \mid \operatorname{rank} A = k, \operatorname{rank} B = l\}.$$

Then $\mathfrak{N}_{k,l}$ is a locally closed set of $\mathfrak{N}$, and it is clearly stable under the action of $K_{\mathbb{C}} \times M'_{\mathbb{C}}$.

Lemma 2.2 $\mathfrak{N}_{k,l}$ *is an orbit of* $K_{\mathbb{C}}^{\circ} \times M'_{\mathbb{C}}$*, which is also stable under the action of* $K_{\mathbb{C}}$*. The dimension of* $\mathfrak{N}_{k,l}$ *is give by*

$$\dim \mathfrak{N}_{k,l} = n(2n - (k+l)) + kp + lq - \frac{k(k+1)}{2} - \frac{l(l+1)}{2}.$$

Proof. Let $(A,B) \in \mathfrak{N}_{k,l}$. Then, since ${}^tAA = 0$, the column vectors of the matrix $A \in M_{p,n}$ generates an isotropic space in $\mathbb{C}^p$ of dimension $k < p/2$. By the action of GL_n from the right, we can assume that the first k-columns of A are non-zero and linearly independent, and the rest is all zero. Let us fix a maximally isotropic space in $\mathbb{C}^p$ for the symmetric bilinear form which defines O_p. Take an arbitrary basis for the space, and write it as $\{e_1, \dots, e_{[p/2]}\}$. Then it is easy to see that the action of SO_p brings the first k-columns of A to $\{e_1, \dots, e_k\}$. This means that an $SO_p \times GL_n$-orbit through A has a representative $\Xi_k = [\ e_1\ e_2\ \cdots\ e_k\ 0\ \cdots 0\ ] \in M_{p,n}$. Similarly, an $SO_q \times GL_n$-orbit through B has a representative $\Xi'_l = [\ e'_1\ e'_2\ \cdots\ e'_l\ 0\ \cdots\ 0\ ] \in M_{q,n}$, where $\{e'_1, \dots, e'_{[q/2]}\}$ is a fixed basis of a maximally totally isotropic space for O_q. Now it is clear that $\mathfrak{N}_{k,l}$ is a single $K^\circ_{\mathbb{C}} \times M'_{\mathbb{C}}$-orbit thorough (Ξ_k, Ξ'_l).

Now the dimension of the orbit is easy to calculate. We omit it. Q.E.D.

From this lemma, we can immediately prove the following corollary, which is the statement (2) of Theorem 2.1.

Corollary 2.3 $\mathfrak{N}$ *is of complete intersection. In particular, it is Cohen-Macaulay.*

Proof. Let $\{C_{ij} \mid 1 \le i \le j \le n\}$ and $\{D_{ij} \mid 1 \le i \le j \le n\}$ be a coordinate system of $\mathfrak{s}' = \mathrm{Sym}_n \oplus \mathrm{Sym}_n$, where we regard C_{ij} and D_{ij} as a linear function on Sym_n which extracts the (i,j)-element of a symmetric matrix. Let

$$\varphi_2^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W]$$

be an algebra homomorphism induced by φ_2. Then $\varphi_2^*(C_{ij})$ and $\varphi_2^*(D_{ij})$ $(1 \le i \le j \le n)$ define the null cone $\mathfrak{N}$. Note that we have an explicit description

$$\varphi_2^*(C_{ij}) = \sum_{k=1}^{p} A_{ki} A_{kj} =: f_{ij},$$
$$\varphi_2^*(D_{ij}) = \sum_{k=1}^{q} B_{ki} B_{kj} =: g_{ij},$$

where A_{ij} (respectively B_{ij}) is a coordinate system on $M_{p,n}$ (respectively $M_{q,n}$). These $\{f_{ij}, g_{ij} \mid 1 \le i \le j \le n\}$ are classical invariants of degree two for O_p and O_q, which generates all the invariants for $K_{\mathbb{C}}$ vanishing at zero.

By Lemma 2.2, we know the codimension of the null cone in W, which is equal to $n(n+1)$. Since the number of defining equations of $\mathfrak{N}$ is also $n(n+1)$, it is a complete intersection. In general, complete intersection implies that its affine regluar function ring is Cohen-Macaulay (see [2, Proposition 18.13] for example). Q.E.D.

The following is also immediate from Lemma 2.2.

Corollary 2.4 *We have the following decomposition of* $\mathfrak{N}$ *by the orbits of* $K_{\mathbb{C}} \times M'_{\mathbb{C}}$ *(or* $K^\circ_{\mathbb{C}} \times M'_{\mathbb{C}}$*):*

$$\mathfrak{N} = \coprod_{0 \le k,l \le n} \mathfrak{N}_{k,l}.$$

Moreover, we have

$$\overline{\mathfrak{N}_{k,l}} = \coprod_{0 \le i \le k,\ 0 \le j \le l} \mathfrak{N}_{i,j}.$$

This corollary will prove the statement (1) of Theorem 2.1.

Let us prove (3) of Theorem 2.1, which is the most difficult part of the theorem. We use the following theorem quoted from [1, Theorem 2.2.11] which is attributed to Bernstein-Lunts, and Kunz.

Theorem 2.5 *Let I be an indeal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by $\{f_1, \ldots, f_p\}$, and put $X = \mathbf{V}(I)$, the variety defined by I. Assume that*

(1) *$\mathbb{C}[x_1, \ldots, x_n]/I$ is Cohen-Macaulay.*

(2) *The differentials $\{df_1, \ldots, df_p\}$ are linearly independent at each point of a certain dense open subset X' of X.*

Then I is equal to the definig radical ideal $\mathbf{I}(X)$, hence $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I$ holds. Moreover, if X is irreducible and $\operatorname{codim}(X - X') \geq 2$, then X is normal.

Let $I = \mathcal{I}^{K_{\mathbb{C}}}$ be an indeal of $\mathbb{C}[W]$ generated by the invariants (without constant term) of $K_{\mathbb{C}}$. It is well-known that $\mathcal{I}^{K_{\mathbb{C}}}$ is generated by the second degree invariants $\{f_{ij} = \varphi_2^*(C_{ij}) \mid 1 \leq i \leq j \leq n\}$ and $\{g_{ij} = \varphi_2^*(D_{ij}) \mid 1 \leq i \leq j \leq n\}$, where C_{ij} and D_{ij} are given in the proof of Corollary 2.3.

By Corollary 2.3, I satisfies the condition (1). Since the differentials $\{df_{ij}, dg_{ij}\}$ are linearly independent at $(\Xi_n, \Xi'_n) \in \mathfrak{N}_{n,n}$ and the $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-orbit $\mathfrak{N}_{n,n}$ is open dense in $\mathfrak{N}$, the condition (2) also holds for $X' = \mathfrak{N}_{n,n}$.

Note that $\mathfrak{N} = \mathbf{V}(\mathcal{I}^{K_{\mathbb{C}}})$ is irreducible by (1) of Theorem 2.1. We must check the codimension of $\mathfrak{N} - \mathfrak{N}_{n,n}$. By Lemma 2.2, we know

$$\begin{aligned}
\operatorname{codim}(\mathfrak{N} - \mathfrak{N}_{n,n}) &= \min\{\operatorname{codim} \overline{\mathfrak{N}_{k,l}} \mid 1 \leq k, l \leq n\} \\
&= \min\{\operatorname{codim} \overline{\mathfrak{N}_{n,n-1}}, \operatorname{codim} \overline{\mathfrak{N}_{n-1,n}}\} \\
&= \min\{p - 2n, q - 2n\}.
\end{aligned}$$

By the assumption $\min(p, q) \geq 2 + 2n$, we have $\operatorname{codim}(\mathfrak{N} - \mathfrak{N}_{n,n}) \geq 2$. Hence we conclude that $X = \mathfrak{N}$ is normal and $\mathcal{I}^{K_{\mathbb{C}}}$ is prime.

Now Theorem 2.1 is completely proved.

Remark 2.6 If $\min(p, q) > 2n$, it is known that $\mathcal{I}^{K_{\mathbb{C}}}$ is prime. This is an immediate consequence from the fact that the same statement for $\mathfrak{N}^{\pm}$ is valid (see [5, § 3.7.3]), and that the tensor product of integral domains over $\mathbb{C}$ remains an integral domain.

Hereafter, *we always assume that* $\min(p, q) > 2n$ except §6.

3 Harmonics on the null cone

Let $\mathcal{H}(K_{\mathbb{C}})$ be the space of $K_{\mathbb{C}}$-harmonic polynomials in $\mathbb{C}[W]$. Then $\mathbb{C}[W]$ is a direct sum of harmonics and the ideal $\mathcal{I}^{K_{\mathbb{C}}}$ generated by invariants without constant term:

$$\mathbb{C}[W] = \mathcal{H}(K_{\mathbb{C}}) \oplus \mathcal{I}^{K_{\mathbb{C}}}.$$

For the proof, see [1, Proposition 6.3.13]. From this, we immediately have

Theorem 3.1 *If $\min(p,q) > 2n$, the ring of regular functions on the null cone $\mathfrak{N}$ is isomorphic to $\mathcal{H}(K_{\mathbb{C}})$ as a $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-module:*

$$\mathbb{C}[\mathfrak{N}] \simeq \mathcal{H}(K_{\mathbb{C}}).$$

PROOF. By Remark 2.6 we have $\mathbf{I}(\mathfrak{N}) = \mathcal{I}^{K_{\mathbb{C}}}$, so that $\mathbb{C}[\mathfrak{N}] \simeq \mathbb{C}[W]/\mathcal{I}^{K_{\mathbb{C}}} \simeq \mathcal{H}(K_{\mathbb{C}})$. The last equality comes from the remark just before the theorem. Q.E.D.

We would like to clarify the explicit structure of $\mathcal{H}(K_{\mathbb{C}})$. Let

$$\mathcal{P}_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$$

be the set of all partitions of length at most n. For $\lambda \in \mathcal{P}_n$, we denote by $\tau_\lambda = \tau_\lambda^{(n)}$ an irreducible finite dimensional representation of GL_n with highest weight λ. Consequently, τ_λ is a polynomial representation. Similarly, we denote by $\sigma_\lambda^{(p)}$ an irreducible finite dimensional representation of SO_p with highest weight λ. Since $\lambda \in \mathcal{P}_n$ and $2n < p$, an element $h \in O_p - SO_p$ acts on $\sigma_\lambda^{(p)}$ as an outer automorphism which fixes $\sigma_\lambda^{(p)}$. Therefore, the representation $\mathbf{1}_{\mathbb{Z}_2} \otimes \sigma_\lambda^{(p)}$ of $O_p = \mathbb{Z}_2 \ltimes SO_p$ is well-defined. We denote it by the same letter $\sigma_\lambda^{(p)}$ by abuse of notation. Note that $\sigma_\lambda^{(p)}$ is itself an irreducible representation of SO_p.

Lemma 3.2 *We have*

$$\mathbb{C}[\mathfrak{N}^+] \simeq \mathcal{H}^+(O_p) \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)} \boxtimes \tau_\lambda^*$$

as an $O_p \times GL_n$-module, where $\mathcal{H}^+(O_p)$ denotes the space of O_p-harmonic polynomials in $\mathbb{C}[W^+]$. Similarly, we have

$$\mathbb{C}[\mathfrak{N}^-] \simeq \mathcal{H}^-(O_q) \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(q)} \boxtimes \tau_\lambda$$

as an $O_q \times GL_n$-module.

PROOF. For $\mathfrak{N}^+ \subset W^+$, the statement follows from [5]. Note that W^- is a contragredient of the usual action. Therefore we must take a contragredient in the expression of $\mathbb{C}[\mathfrak{N}^-]$, while $(\sigma_\lambda^{(q)})^* \simeq \sigma_\lambda^{(q)}$. Q.E.D.

Theorem 3.3 *We have a multiplicity free decomposition of the regular function ring of $\mathfrak{N}$,*

$$\mathbb{C}[\mathfrak{N}] \simeq \sum_{\lambda, \mu \in \mathcal{P}_n}^{\oplus} (\sigma_\lambda^{(p)} \boxtimes \sigma_\mu^{(q)}) \boxtimes (\tau_\lambda^* \boxtimes \tau_\mu)$$

as a $K_{\mathbb{C}} \times M'_{\mathbb{C}} = (O_p \times O_q) \times (GL_n \times GL_n)$-module. As a result, the action of $K_{\mathbb{C}} \times M'_{\mathbb{C}}$ on $\mathfrak{N}$ is multiplicity free.

PROOF. Since $\mathbb{C}[\mathfrak{N}] \simeq \mathbb{C}[\mathfrak{N}^+] \otimes \mathbb{C}[\mathfrak{N}^-]$, the satement follows from Lemma 3.2. Q.E.D.

From this theorem, we see that the null cone $\mathfrak{N}$ with the action of $K_{\mathbb{C}} \times M'_{\mathbb{C}}$ is an example of the variety X in Sato's paper [7] with $\mathbf{G}_1 = M'_{\mathbb{C}} \supset \mathbf{H} = K'_{\mathbb{C}}$ and $\mathbf{G}_2 = K_{\mathbb{C}}$, where the boldface notation is adopted from Sato's paper.

We use the notation from [6, § 8] and study the Poincare series and degrees of $K'_{\mathbb{C}}$-covariants. Recall that $M'_{\mathbb{C}} = GL_n \times GL_n \supset K'_{\mathbb{C}} = \Delta GL_n$. From Theorem 3.3, we know the ring of $K'_{\mathbb{C}}$-invariants is decomposed as follows

$$\Gamma(\mathfrak{N}; \mathbf{1}_{K'_{\mathbb{C}}}) := \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}} \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)} \boxtimes \sigma_\lambda^{(q)} \quad (\text{as a } K_{\mathbb{C}} = O_p \times O_q\text{-module}). \tag{3.1}$$

Therefore, the Poincaré series of the invariants is given by

$$P(\mathbf{1}_{K'_{\mathbb{C}}}; t) = \sum_{\lambda \in \mathcal{P}_n} \dim(\sigma_\lambda^{(p)} \boxtimes \sigma_\lambda^{(q)})\, t^{2|\lambda|},$$

where $|\lambda| = \sum_{i=1}^n \lambda_i$ denotes the usual size function. Since the $K'_{\mathbb{C}}$-invariants are generated by degree two polynomials, we have

$$P(\mathbf{1}_{K'_{\mathbb{C}}}; t) = \frac{Q(t)}{(1-t^2)^d},$$

where d is the transcendence degree of the quotient field of $\mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}}$ over $\mathbb{C}$ (hence, it is a positive integer), and $Q(t)$ is a polynomial in t which does not vanish at $t = 1$. We call $Q(1)$ the degree of invariants, and denote $\operatorname{Deg} \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}} := Q(1)$. It is known that the degree is a positive integer. We shall return back to these formulas afterwards.

Next, let us examine covariants. Let τ_ν be an irreducible representation of GL_n. Note that ν need not be a partition but a dominant integral weight. Let us denote by Φ_n^+ the set of dominant integral weights for GL_n. For $\lambda, \mu \in \Phi_n^+$, decompose $\tau_\lambda \otimes \tau_\mu$ as a GL_n-module

$$\tau_\lambda \otimes \tau_\mu = \sum_{\nu \in \Phi_n^+} c_{\lambda\,\mu}^{\,\nu} \tau_\nu,$$

where the multiplicity $c_{\lambda\,\mu}^{\,\nu}$ is called the Littlewood-Richardson coefficient. Let us denote the highest weight of τ_μ^* by μ^*. Then we have a formula for τ_ν-covariants on $\mathfrak{N}$:

$$\Gamma(\mathfrak{N}; \tau_\nu) \simeq \sum_{\lambda, \mu \in \mathcal{P}_n}^{\oplus} c_{\lambda^*\,\mu}^{\,\nu} \sigma_\lambda^{(p)} \boxtimes \sigma_\mu^{(q)},$$

as a $K_{\mathbb{C}} = O_p \times O_q$-module. Its Poincare series is given by

$$P(\tau_\nu; t) = \sum_{\lambda, \mu \in \mathcal{P}_n} c_{\lambda^*\,\mu}^{\,\nu} \dim(\sigma_\lambda^{(p)} \boxtimes \sigma_\mu^{(q)})\, t^{|\lambda| + |\mu|}.$$

Again, $P(\tau_\nu; t)$ can be expressed as

$$P(\tau_\nu; t) = \frac{Q_{\tau_\nu}(t)}{(1-t^2)^d}$$

with the same d and a certain polynomial $Q_{\tau_\nu}(t)$. We denote $\operatorname{Deg}\Gamma(\mathfrak{N}; \tau_\nu) = Q_{\tau_\nu}(1)$ and call it the degree of covariants $\Gamma(\mathfrak{N}; \tau_\nu)$. The degree of covariants is a non-negative integer in general. Moreover, we have

Theorem 3.4 *The degree of covariants is given by*

$$\operatorname{Deg}\Gamma(\mathfrak{N}; \tau_\nu) = \dim \tau_\nu \operatorname{Deg} \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}}.$$

PROOF. We can explicitly check the condition (1) – (3) in [6, § 8.2]. In [6, § 8], the variety X is assumed to be a vector space, however, all the arguments is valid in the case of the null cone. Now the statement follows from [6, Theorem 8.6]. Q.E.D.

4 Contraction of the null cone

In this section, we study the image of the null cone by the map φ_1. Namely, put

$$\mathfrak{X} = \varphi_1(\varphi_2^{-1}(0)) = \{A\,{}^tB \in M_{p,q} \mid (A, B) \in W, \text{ and } {}^tAA = 0, {}^tBB = 0\}.$$

We call $\mathfrak{X}$ the *contraction* of the null cone $\mathfrak{N}$. Since φ_1 is $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant and $\mathfrak{N}$ is $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-stable, $\mathfrak{X}$ is stable under the action of $K_{\mathbb{C}}$.

Theorem 4.1 *The contraction $\mathfrak{X}$ of the null cone $\mathfrak{N}$ is the closure of a single $K_{\mathbb{C}} = O_p \times O_q$-orbit. We denote the orbit by $\mathcal{O}_n$ so that $\mathfrak{X} = \overline{\mathcal{O}_n}$. The orbital decomposition of $\mathfrak{X}$ is given by*

$$\mathfrak{X} = \overline{\mathcal{O}_n} = \coprod_{0 \le k \le n} \mathcal{O}_k.$$

PROOF. Tentatively, we put

$$\mathfrak{X}_0 = \{X \in M_{p,q} \mid X\,{}^tX = 0, {}^tXX = 0, \operatorname{rank} X \le n\} \supset \mathfrak{X}.$$

We shall prove that $\mathfrak{X}_0 = \mathfrak{X}$, hence we can conclude that $\mathfrak{X}$ is Zariski closed.

It is convenient to work with the symmetric bilinear form induced by

$$J_p = \begin{bmatrix} 0 & 1_m \\ 1_m & 0 \end{bmatrix},$$

where $p = 2m$ is even, or

$$J_p = \begin{bmatrix} J_{2m} & 0 \\ 0 & 1 \end{bmatrix},$$

where $p = 2m+1$ is odd. If we consider these symmetric bilinear forms, we shall denote $O_p^J, \mathfrak{X}^J$, etc., with the superscript J. Put

$$c_p = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_m & 1_m \\ \sqrt{-1}\, 1_m & -\sqrt{-1}\, 1_m \end{bmatrix},$$

where $p = 2m$ is even, or

$$c_p = \begin{bmatrix} c_{2m} & 0 \\ 0 & 1 \end{bmatrix},$$

where $p = 2m+1$ is odd. Then it is easy to verify that ${}^t c_p c_p = J_p$ and

$$O_p^J \ni g \longmapsto c_p g c_p^{-1} \in O_p$$

gives an isomorphism. Also we have

$$\begin{aligned} \mathfrak{X}^J &= \{ J_p A\, {}^t B J_q \mid (A, B) \in \mathfrak{N} \}, \qquad \text{and} \\ \mathfrak{X}_0^J &= \{ X \in M_{p,q}^J \mid X J_q\, {}^t X = 0,\ {}^t X J_p X = 0, \operatorname{rank} X \le n \}, \end{aligned}$$

where $M_{p,q}^J = M_{p,q}$ as a vector space, but with the different action of orthogonal groups. Namely, the action of $O_p^J \times O_q^J$ on $M_{p,q}^J$ is equivalent to that of $O_p \times O_q$ on $M_{p,q}$ via the map

$$M_{p,q}^J \ni X \longmapsto c_p X\, {}^t c_q \in M_{p,q}.$$

Now we turn to the proof. First, note that the column vectors of $X \in \mathfrak{X}_0^J$ generates a totally isotropic space in $\mathbb{C}^p$. Since $k := \operatorname{rank} X \le n$, there exists a $g \in O_p^J$ such that

$$gX = \begin{bmatrix} 1_k & * \\ 0 & 0 \end{bmatrix}.$$

The condition $X J_q\, {}^t X = 0$ tells as that the row vectors generates a totally isotropic space in $\mathbb{C}^q$ as well. Since the row vectros of gX is already linearly independent, we can find an $h \in O_q^J$ such that

$$gXh = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} =: \Xi_k^J.$$

This shows that $\mathfrak{X}_0^J$ is a union of orbits through Ξ_k^J $(0 \le k \le n)$. However, it is easy to see that $\Xi_k^J \in \mathfrak{X}^J$ for all k. This means that $\mathfrak{X}_0 = \mathfrak{X}$ and the theorem follows. We remark that $\mathcal{O}_k$ is a $K_{\mathbb{C}}$-orbit through $\Xi_k = c_p^{-1} \Xi_k^J\, {}^t c_p^{-1}$. Q.E.D.

Corollary 4.2 *The fiber of* $\overline{\mathcal{O}_k}$ $(0 \le k \le n)$ *is given by*

$$\varphi_1^{-1}(\overline{\mathcal{O}_k}) = \coprod_{0 \le \min(i,j) \le k} \mathfrak{N}_{i,j}.$$

PROOF. Compare the rank, and we have the result. Q.E.D.

Lemma 4.3 *For $0 \le k \le n$, we have*

$$\dim \overline{\mathcal{O}_k} = k((p+q) - (2k+1)).$$

PROOF. The Lie algebra of O^J_p is given by

$$\mathfrak{o}^J_p = \left\{ X = \left[\begin{array}{c|c} a & b \\ \hline c & -{}^t a \end{array} \right] \mid a \in \mathfrak{gl}_m, b, c \in \operatorname{Alt}_m \right\},$$

if $p = 2m$ is even, and

$$\mathfrak{o}^J_p = \left\{ X = \left[\begin{array}{c|c} Y & b \\ \hline -{}^t b J_{2m} & 0 \end{array} \right] \mid Y \in \mathfrak{o}^J_{2m}, b \in \mathbb{C}^{2m} \right\},$$

if $p = 2m + 1$ is odd.

For simplicity, we assume that both p and q are even. Put $p = 2m$ and $q = 2l$. Then, an element $(X, Y) \in \mathfrak{o}^J_p \oplus \mathfrak{o}^J_q$ which kills Ξ^J_k is of the following form

$$(X, Y) = \left(\left[\begin{array}{c|c} A & B \\ \hline C & -{}^t A \end{array} \right], \left[\begin{array}{c|c} D & E \\ \hline F & -{}^t D \end{array} \right] \right);$$
$$A = \begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \ C = \begin{bmatrix} 0_k & 0 \\ 0 & * \end{bmatrix}, \ D = \begin{bmatrix} -{}^t a & * \\ 0 & * \end{bmatrix}, \ F = \begin{bmatrix} 0_k & 0 \\ 0 & * \end{bmatrix},$$

where $a \in M_{k,k}$ and $B \in \operatorname{Alt}_m, E \in \operatorname{Alt}_l$ are arbitrary.

Now it is easy to count the dimension. Q.E.D.

5 Geometry of orbits

In this section, we invetigate the regular function ring of $\overline{\mathcal{O}_n}$ ($0 \le 2n < \min(p, q)$) and prove the normality of $\overline{\mathcal{O}_n}$.

Let $\varphi_1^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[\mathfrak{N}]$ be an algebra morphism induced by the map $\varphi_1 : \mathfrak{N} \longrightarrow \mathfrak{s}$. Since the image $\varphi_1(\mathfrak{N})$ is $\overline{\mathcal{O}_n}$, it actually induces an injection

$$\varphi_1^* : \mathbb{C}[\overline{\mathcal{O}_n}] \longrightarrow \mathbb{C}[\mathfrak{N}], \tag{5.1}$$

which is a graded morphism of degree two. Recall the action of $K'_{\mathbb{C}} = \Delta GL_n$ on $\mathfrak{N}$.

Theorem 5.1 *The image of φ_1^* in (5.1) coincides with the $K'_{\mathbb{C}}$-invariants in $\mathbb{C}[\mathfrak{N}]$, i.e.,*

$$\varphi_1^* : \mathbb{C}[\overline{\mathcal{O}_n}] \xrightarrow{\ \sim\ } \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}}.$$

PROOF. Look at the original map $\varphi_1 : W \longrightarrow \mathfrak{s}$. Then the induced map $\varphi_1^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[W]$ is given by

$$\varphi_1^*(X_{ij}) = \sum_{k=1}^{n} A_{ik}B_{jk} =: F_{ij},$$

where $\{X_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ is the set of coordinates on $\mathfrak{s} = M_{p,q}$. By the classical invariant theory, it is known that $\{F_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ generates the ring of $K'_{\mathbb{C}}$-invariants in $\mathbb{C}[W]$, so that $\operatorname{Image}\varphi_1^* = \mathbb{C}[W]^{K'_{\mathbb{C}}}$.

Now consider the restriction $\varphi_1 : \mathfrak{N} \longrightarrow \mathfrak{s}$, which we denote by the same notation. By the above consideration, we know that the induced map gives actually an algebra morphism

$$\varphi_1^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[W]^{K'_{\mathbb{C}}}/\mathbf{I}(\mathfrak{N})^{K'_{\mathbb{C}}} \subset \mathbb{C}[\mathfrak{N}] = \mathbb{C}[W]/\mathbf{I}(\mathfrak{N}).$$

This morphism is surjective by definition. Now, since $\mathbb{C}[\mathfrak{s}]/\operatorname{Ker}\varphi_1^* = \mathbb{C}[\overline{\mathcal{O}_n}]$ by Theorem 4.1, we conclude the theorem. Q.E.D.

Corollary 5.2 *If $\mathfrak{N}$ is normal, $\overline{\mathcal{O}_n}$ is also a normal variety. In particular, if* $\min(p,q) \geq 2 + 2n$*, the closure $\overline{\mathcal{O}_n}$ is an irreducible normal affine variety.*

PROOF. Theorem 5.1 says that $\overline{\mathcal{O}_n}$ is a geometric quotient of $\mathfrak{N}$ by $K'_{\mathbb{C}}$. Since a geometric quotient of a normal variety is also normal, the corollary follows. Q.E.D.

Corollary 5.3 *As a $K_{\mathbb{C}} = O_p \times O_q$-module, we have*

$$\mathbb{C}[\overline{\mathcal{O}_n}] \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)} \boxtimes \sigma_\lambda^{(q)}.$$

Moreover, this is equal to the global section $\mathbb{C}[\mathcal{O}_n]$.

PROOF. The first statement follows from Theorem 5.1 and (3.1). Since

$$\dim \mathcal{O}_n - \dim \mathcal{O}_{n-1} = p + q + 1 - 4n \geq 2$$

by Lemma 4.3, we conclude $\mathbb{C}[\overline{\mathcal{O}_n}] = \mathbb{C}[\mathcal{O}_n]$. Q.E.D.

Corollary 5.4 *We have*

$$\operatorname{Deg} \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}} = \deg \overline{\mathcal{O}_n}, \qquad \textit{and}$$
$$\text{trans.deg}_{\mathbb{C}}\mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}} = \dim \overline{\mathcal{O}_n} = n((p+q) - (2n+1)),$$

where $\deg \overline{\mathcal{O}_n}$ denotes the degree of a projective variety (note that $\overline{\mathcal{O}_n}$ is an affine cone).

PROOF. These equations follow from the definition of ring theoretic degrees and geometric degrees. Q.E.D.

6 Reversed correspondence

Now we briefly summarize what we get if we lift the trivial orbit of $\mathfrak{s} = M_{p,q}$ to that of $\mathfrak{s}' = \mathrm{Sym}_n \oplus \mathrm{Sym}_n$.

Caution: we assume $p+q<n$ in this section.

Let us define the null cone $\mathfrak{N}'$ by

$$\mathfrak{N}' = \varphi_1^{-1}(0) = \{(A,B) \in W \mid A\,{}^tB = 0\}.$$

Put

$$\Xi_k = \begin{bmatrix} 1_k & \\ & 0 \end{bmatrix} \in M_{p,n} \in W^+, \quad \Upsilon_l = \begin{bmatrix} 0 & \\ & 1_l \end{bmatrix} \in M_{q,n} \in W^-,$$

and denote the $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit through $(\Xi_k, \Upsilon_l) \in W$ by $\mathfrak{N}'_{k,l}$. Recall that $M_{\mathbb{C}} = GL_p \times GL_q$ and $K'_{\mathbb{C}} = \Delta GL_n$.

Proposition 6.1 *We assume $p+q<n$.*

(1) *The null cone $\mathfrak{N}'$ is the closure of a single $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\mathfrak{N}'_{p,q}$ in W. In particular, $\mathfrak{N}'$ is an irreducible affine variety.*

(2) *We have the following decomposition of $\mathfrak{N}'$ by the orbits of $M_{\mathbb{C}} \times K'_{\mathbb{C}}$:*

$$\mathfrak{N}' = \coprod_{0 \le k \le p, 0 \le l \le q} \mathfrak{N}'_{k,l}.$$

Moreover, we have

$$\overline{\mathfrak{N}'_{k,l}} = \coprod_{0 \le i \le k, 0 \le j \le l} \mathfrak{N}'_{i,j}.$$

(3) *$\mathfrak{N}'$ has the codimension pq in W. More generally, the dimension of $\mathfrak{N}'_{k,l}$ is given by*

$$\dim \mathfrak{N}'_{k,l} = k(n+p-k) + l(n+q-l) - kl.$$

Theorem 6.2 *We assume $p+q<n$. Then the null cone $\mathfrak{N}'$ is a variety of complete intersection, and $\dim \mathfrak{N}' = n(p+q) - pq$. In particular, the regular function ring $\mathbb{C}[\mathfrak{N}']$ is Cohen-Macaulay. Moreover, $\mathfrak{N}'$ is an irreducible normal variety.*

Theorem 6.3 *If $p+q<n$, the ring of regular functions on the null cone $\mathfrak{N}'$ is isomorphic to $\mathcal{H}(K'_{\mathbb{C}})$ as a $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-module:*

$$\mathbb{C}[\mathfrak{N}'] \simeq \mathcal{H}(K'_{\mathbb{C}}).$$

Take two partitions $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathcal{P}_p$ and $\mu = (\mu_1, \ldots, \mu_p) \in \mathcal{P}_q$. We define a dominant integral weight $\lambda \odot \mu$ for GL_n by

$$\lambda \odot \mu = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\mu_q, \ldots, -\mu_1) \in \Phi_n^+,$$

and call it a *composit* of λ and μ.

Theorem 6.4 *We have a multiplicity free decomposition of the regular function ring of $\mathfrak{N}'$,*

$$\mathbb{C}[\mathfrak{N}'] \simeq \sum_{\lambda\in\mathcal{P}_p,\mu\in\mathcal{P}_q}^{\oplus} \left((\tau_\lambda^{(p)})^* \boxtimes \tau_\mu^{(q)}\right) \boxtimes (\tau_{\lambda\odot\mu}^{(n)})^* \simeq \sum_{\lambda\in\mathcal{P}_p,\mu\in\mathcal{P}_q}^{\oplus} \left((\tau_\lambda^{(p)})^* \boxtimes \tau_\mu^{(q)}\right) \boxtimes \tau_{\mu\odot\lambda}^{(n)}$$

as a $M_{\mathbb{C}} \times K'_{\mathbb{C}} = (GL_p \times GL_q) \times \Delta GL_n$-module. As a result, the action of $M_{\mathbb{C}} \times K'_{\mathbb{C}}$ on $\mathfrak{N}'$ is multiplicity free.

We define a contraction of the null cone by

$$\mathfrak{Y} = \varphi_2(\mathfrak{N}') = \{({}^tAA, {}^tBB) \in \mathfrak{s}' \mid (A, B) \in W, A\,{}^tB = 0\}.$$

Theorem 6.5 *The contraction $\mathfrak{Y}$ of the null cone $\mathfrak{N}'$ is the closure of a single $K'_{\mathbb{C}} = \Delta GL_n$-orbit through*

$$\left(\begin{bmatrix} 1_p & \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \\ & 1_q \end{bmatrix}\right) \in \mathfrak{s}' = \mathrm{Sym}_n \oplus \mathrm{Sym}_n.$$

We denote the orbit by $\mathcal{O}_{p,q}$ so that $\mathfrak{Y} = \overline{\mathcal{O}_{p,q}}$.

(1) *The orbital decomposition of $\mathfrak{Y}$ is given by*

$$\mathfrak{Y} = \overline{\mathcal{O}_{p,q}} = \coprod_{0\le k\le p, 0\le l\le q} \mathcal{O}_{k,l}.$$

(2) *The fiber of $\overline{\mathcal{O}_{k,l}}$ $(k \le p, l \le q)$ is given by*

$$\varphi_2^{-1}(\overline{\mathcal{O}_{k,l}}) = \overline{\mathfrak{N}'_{k,l}}.$$

(3) *For $p + q < n$, the dimension of the orbit is given by*

$$\dim \overline{\mathcal{O}_{p,q}} = p\left(n - \frac{p-1}{2}\right) + q\left(n - \frac{q-1}{2}\right) - pq.$$

Theorem 6.6 *The image of $\varphi_2^* : \mathbb{C}[\overline{\mathcal{O}_{p,q}}] \longrightarrow \mathbb{C}[\mathfrak{N}']$ coincides with the $K_{\mathbb{C}}$-invariants in $\mathbb{C}[\mathfrak{N}']$, i.e.,*

$$\varphi_2^* : \mathbb{C}[\overline{\mathcal{O}_{p,q}}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{N}']^{K_{\mathbb{C}}},$$

where $K_{\mathbb{C}} = O_p \times O_q$.

Corollary 6.7 *If $p+q<n$, the closure $\overline{\mathcal{O}_{p,q}}$ is an irreducible normal affine variety.*

Corollary 6.8 *We assume that $p+q<n$. As a $K'_{\mathbb{C}} \simeq GL_n$-module, the regular function ring decompses as*

$$\mathbb{C}[\overline{\mathcal{O}_{p,q}}] \simeq \sum_{\lambda \in \mathcal{P}_p, \mu \in \mathcal{P}_q}^{\oplus} (\tau^{(n)}_{2\lambda \odot 2\mu})^{*}. \simeq \sum_{\lambda \in \mathcal{P}_p, \mu \in \mathcal{P}_q}^{\oplus} \tau^{(n)}_{2\mu \odot 2\lambda}.$$

Moreover, this is equal to the global section $\mathbb{C}[\mathcal{O}_{p,q}]$.

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