THETA LIFTING OF NILPOTENT ORBITS FOR SYMMETRIC PAIRS
A long version

KYO NISHIYAMA AND CHEN-BO ZHU

Abstract. We consider a reductive dual pair (G, G') in the stable range with G' the smaller member and of Hermitian symmetric type. We study theta lifting of nilpotent $K'_{\mathbb{C}}$-orbits, where K' is a maximal compact subgroup of G'. The main result is the description of the precise $K_{\mathbb{C}}$-module structure of the regular function ring of the closure of the lifted nilpotent orbit of the symmetric pair (G, K). As an application, we prove sphericality and normality of the closure of certain nilpotent $K_{\mathbb{C}}$-orbits obtained this way.

Contents

Date: Ver. 1.0 [Wed Jul 17 10:06:19 JST 2002] (compiled on July 17, 2002).
1991 *Mathematics Subject Classification.* Primary 22E46, 11F27.
Key words and phrases. reductive dual pair, theta lifting, nilpotent orbits, harmonic polynomial, invariant theory.

Introduction

Let (G, G') be a reductive dual pair in a symplectic group $\mathbb{G} = Sp(2N, \mathbb{R})$, where N denotes the rank of $\mathbb{G}$. Throughout this paper, we will assume that (G, G') is of type I, and it is in the stable range with G' the smaller member (cf. [5, 11]). We will also assume that G'/K' is an irreducible Hermitian symmetric space. By the classification of irreducible dual pairs, our restriction amounts to saying that (G, G') is from the following list (cf. [7]).

Table 1. The dual pairs treated in this paper

the pair (G, G')	stable range condition
$(O(p,q), Sp(2n,\mathbb{R}))$	$2n < \min(p,q)$
$(U(p,q), U(m,n)))$	$m+n \leq \min(p,q)$
$(Sp(p,q), O^*(2n))$	$n \leq \min(p,q)+1$

Note that we have excluded the equality $2n = \min(p, q)$ in the first case, to avoid some (small) technicalities.

▶**[Extra Remarks]** This condition is equivalent to the condition that all the constituents are holomorphic discrete series for the compact dual pairs $K^{\pm} \times G'$. ◀

We introduce some notations. Let $\mathfrak{g}$ (resp. $\mathfrak{g}'$) be the complexification of the Lie algebra of G (resp. G') and K (resp. K') a maximal compact subgroup of G (resp. G'). Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \qquad \mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}',$$

be respectively Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{g}'$, compatible in certain way (see §2).

Let $\mathbb{O}' \subset \mathfrak{s}'$ be a nilpotent $K'_{\mathbb{C}}$-orbit. Then using certain double fiberation from $W \simeq \mathbb{C}^N$, we can define the *theta lift* of $\mathbb{O}'$, which is a certain nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}$ in $\mathfrak{s}$ (see §2; cf. [19]). We shall denote $\mathbb{O} = \theta(\mathbb{O}')$. Let $\mathbb{C}[\overline{\mathbb{O}}]$ (resp. $\mathbb{C}[\overline{\mathbb{O}'}]$) be the regular function ring on the closure of $\mathbb{O}$ (resp. $\mathbb{O}'$). The purpose of this paper is to investigate the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathbb{O}}]$ in terms of the $K'_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathbb{O}'}]$. We will show that the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathbb{O}}]$ can in fact be described entirely and explicitly by the regular function ring $\mathbb{C}[\overline{\mathbb{O}'}]$ and the space of $K_{\mathbb{C}}$-harmonic polynomials $\mathcal{H}(K_{\mathbb{C}})$ on W (using certain branching coefficients). The main results are summarized in Theorem 2.9.

In §3, we will give some examples of the lifted orbits and their regular function rings. As an application we prove normality of the closure of some of these nilpotent orbits, which are lifted from the smaller group. Also, we obtain a family of spherical nilpotent orbits. We believe that they occupy a large portion of the set of all the spherical nilpotent $K_{\mathbb{C}}$-orbits, which are not classified yet.

In the Appendix, we describe the explicit form of moment maps and give some basic properties of null cones. Most of the results in the Appendix are known to the experts, but we include them because of lack of appropriate references.

1. Diamond pairs

We review certain structure results related to our dual pairs [7].

Let $W_{\mathbb{R}} \simeq \mathbb{R}^{2N}$ be a real symplectic space which realizes $\mathbb{G} = Sp(2N, \mathbb{R})$ as a symplectic group on $W_{\mathbb{R}}$. There is a canonical complex structure on $W_{\mathbb{R}}$ and we can view $W_{\mathbb{R}}$ as (the underlying real vector space of) a complex vector space $W \simeq \mathbb{C}^N$. The symplectic form on $W_{\mathbb{R}}$ is then given by the imaginary part of a canonical positive definite Hermitian form on W. By this identification, a maximal compact subgroup $\mathbb{K}$ of $\mathbb{G}$ is realized as the unitary group $U(W) \simeq U(N)$ on $W \simeq \mathbb{C}^N$.

We may choose maximal compact subgroups K and K' of G and G' respectively, in such a way that $K \cdot K'$ is contained in the standard maximal compact subgroup $\mathbb{K} \simeq U(N)$ of $\mathbb{G}$.

In view of Table 1, we sometimes write $G = G(p, q)$, namely $G(p, q)$ will denote one of the groups $O(p, q), U(p, q)$ or $Sp(p, q)$. We denote simply $G(k) = G(k, 0)$, which is compact. Though we are mainly concerned with *non-compact* cases, results on compact cases will be used heavily.

From the pair (G, G'), we define another three dual pairs, which form so-called *diamond dual pairs* (see [7, §5]). Namely, take the commutant of K in $\mathbb{G}$ and denote it by M'. Then M' is also of Hermitian symmetric type and isomorphic to $G' \times G'$ containing G' as the diagonal. The pair (K, M') is a dual pair of compact type. Also, take the full commutant of K in $\mathbb{K}$, and denote it by L'. Then L' is a maximal compact subgroup of M', and it is isomorphic to $K' \times K'$ and contains K' as a diagonal subgroup. Similarly, define M as the commutant of K' in $\mathbb{G}$, and L the commutant in $\mathbb{K}$.

Let us summarize this somewhat complicated situation by the following diagram (Fig. 1).

An explicit description of the diamond pairs for our three cases is given in [7, (5.3)]. In the table there, L (resp. L') in our notation is written as $M^{(1,1)}$ (resp. $M'^{(1,1)}$). For convenience of readers, we reproduce the table here (Table 2 below) with some additional remarks.

Note that M/L is a Hermitian symmetric space (not necessarily irreducible), and the containment $M \supset G$ (resp. $L \supset K$) may or may not be the diagonal map, but it is always a symmetric pair.

Recall $\mathbb{G} = Sp(W_{\mathbb{R}})$ and the complex vector space $W \simeq \mathbb{C}^N$ which is identical to $W_{\mathbb{R}}$ as a real vector space. Then there exists a direct sum decomposition $W_{\mathbb{R}} = W_{\mathbb{R}}^+ \oplus W_{\mathbb{R}}^-$ and correspondingly $W = W^+ \oplus W^-$, which are compactible with direct product decompositions $L = L^+ \times L^-$ and $K = K^+ \times K^-$ in the following way. The subgroups

FIGURE 1. The diagram of a diamond pair.

$$\begin{tikzcd}
& M \arrow[dl, dash] \arrow[dr, dash] \arrow[rrrr, leftrightarrow] & & & & K' \arrow[dl, dash] \arrow[dr, dash, "\Delta"] & \\
L = L^+ \times L^- \arrow[dr, dash] & & G \arrow[dl, dash] \arrow[rr, leftrightarrow] & & G' \arrow[dr, dash, "\Delta"] & & L' = K' \times K' \arrow[dl, dash] \\
& K = K^+ \times K^- \arrow[rrrr, leftrightarrow] & & & & M' = G' \times G' &
\end{tikzcd}$$

TABLE 2. Three diamond dual pairs.

(G, G')	M	$K = K^+ \times K^-$	$L = L^+ \times L^-$	K'
$(O(p,q), Sp(2n,\mathbb{R}))$	$U(p,q)$	$O(p)\times O(q)$	$U(p)\times U(q)$	$U(n)$
$(U(p,q), U(m,n))$	$U(p,q)\times U(p,q)$	$U(p)\times U(q)$	$(U(p)\times U(p))\times(U(q)\times U(q))$	$U(m)\times U(n)$
$(Sp(p,q), O^*(2n))$	$U(2p,2q)$	$Sp(p)\times Sp(q)$	$U(2p)\times U(2q)$	$U(n)$

$$M' = G' \times G', \qquad L' = K' \times K'$$

$L^\pm$ and $K^\pm$ are contained in the unitary group $U(W^\pm)$. Moreover the pairs $(K^\pm, G')$ and $(L^\pm, K')$ are dual pairs in $Sp(W^\pm_{\mathbb{R}})$. Also note that $L^\pm \supset K^\pm$ is a symmetric pair.

For explicit realization of these decompositions, see the Appendix.

2. THETA LIFT OF A NILPOTENT ORBIT

Let $\mathfrak{G} = \mathrm{Lie}\,(\mathbb{G})_{\mathbb{C}}$ (resp. $\mathfrak{K} = \mathrm{Lie}\,(\mathbb{K})_{\mathbb{C}}$) be the complexified Lie algebra of $\mathbb{G} = Sp(2N,\mathbb{R})$ (resp. $\mathbb{K} = U(N)$). Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be the corresponding (complexified) Cartan decomposition. Note that the complexified Lie group $\mathbb{K}_{\mathbb{C}}$ acts on the Cartan space $\mathfrak{P}$ by the restriction of the adjoint action, and this action breaks $\mathfrak{P}$ into $\mathfrak{P}_\pm$, each of which can be identified with a copy of the space $\mathrm{Sym}_N(\mathbb{C})$ of complex symmetric matrices of size N. The action of $k \in GL(N,\mathbb{C}) \simeq \mathbb{K}_{\mathbb{C}}$ on $\mathfrak{P} = \mathfrak{P}_+ \oplus \mathfrak{P}_- \simeq \mathrm{Sym}_N(\mathbb{C}) \oplus \mathrm{Sym}_N(\mathbb{C})$ is given by

$$k \cdot (X, Y) = (kX\,{}^t k,\ {}^t k^{-1} Y k^{-1}) \qquad (X, Y \in \mathrm{Sym}_N(\mathbb{C})).$$

Let $\mathbb{O}_{\min} \subseteq \mathfrak{P}_+$ be the minimal nilpotent $\mathbb{K}_{\mathbb{C}} \simeq GL(N,\mathbb{C})$ orbit in $\mathfrak{P}_+$. We have the identification $\mathbb{O}_{\min} = \{(X,0) \mid X \in \mathrm{Sym}_N(\mathbb{C}), \mathrm{rank}\, X = 1\}$. It is also well-known that there is a map

$$\begin{array}{rccll} j: & W & \longrightarrow & \overline{\mathbb{O}_{\min}} = \mathbb{O}_{\min} \cup \{0\}, & \\ & w & \longmapsto & ({}^t ww, 0), & \qquad w = (w_1, \ldots, w_N) \end{array}$$

which gives $\overline{\mathbb{O}_{\min}}$ as a geometric quotient of the space W by the action of $O(1,\mathbb{C}) = \{\pm 1\}$, and that the quotient map is $\mathbb{K}_{\mathbb{C}}$-equivariant.

Let us consider the dual pair $(G, G') \subseteq \mathbb{G}$. The maximal compact subgroup K (resp. K') determines a Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}'$) of G (resp. G'):

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \qquad \mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}',$$

where $\mathfrak{k}$ (resp. $\mathfrak{k}'$) is the complexified Lie algebra of K (resp. K'), and $\mathfrak{s}$ (resp. $\mathfrak{s}'$) is the orthogonal complement of $\mathfrak{k}$ (resp. $\mathfrak{k}'$) with respect to the Killing form. The subspace $\mathfrak{s}$ (resp. $\mathfrak{s}'$) carries a linear action of $K_{\mathbb{C}}$ (resp. $K'_{\mathbb{C}}$) via the restriction of the adjoint representation. Since G'/K' is a Hermitian symmetric space, $\mathfrak{s}'$ breaks up into irreducible pieces ${\mathfrak{s}'}_{\pm}$ under the action of $K'_{\mathbb{C}}$.

We can arrange the Cartan decompositions so that $\mathfrak{s} \oplus \mathfrak{s}'$ is contained in $\mathfrak{P}$. Then the inclusion map $\mathfrak{s} \subset \mathfrak{P}$ (resp. $\mathfrak{s}' \subset \mathfrak{P}$) induces the projection $\mathfrak{P}^* \to \mathfrak{s}^*$ (resp. $\mathfrak{P}^* \to {\mathfrak{s}'}^*$). By identifying the complex linear dual $\mathfrak{P}^*$ (resp. $\mathfrak{s}^*$, ${\mathfrak{s}'}^*$) with $\mathfrak{P}$ (resp. $\mathfrak{s}$, $\mathfrak{s}'$) by the invariant trace form of matrices, we obtain two projection maps $P_{\mathfrak{s}} : \mathfrak{P} \to \mathfrak{s}$ and $P_{\mathfrak{s}'} : \mathfrak{P} \to \mathfrak{s}'$. The projections restricted to $\overline{\mathbb{O}_{\min}}$ will then induce the so-called moment maps:

$$\begin{array}{l} \varphi : \; W \xrightarrow[]{\substack{\text{geom. quotient} \\ \text{by } \{\pm 1\}}} \overline{\mathbb{O}_{\min}} \xrightarrow{\text{projection}} \mathfrak{s} \\[2ex] \psi : \; W \xrightarrow{\phantom{\text{geom. quotient}}} \overline{\mathbb{O}_{\min}} \xrightarrow{\phantom{\text{projection}}} \mathfrak{s}' \end{array}$$

These maps are $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant with the trivial $K_{\mathbb{C}}$-action on $\mathfrak{s}'$, and the trivial $K'_{\mathbb{C}}$-action on $\mathfrak{s}$ respectively. Also we can (and we will) choose the subspaces ${\mathfrak{s}'}_{\pm} \subset \mathfrak{s}'$ and $W^{\pm} \subset W$ so that

$$\psi(W^{\pm}) \subset {\mathfrak{s}'}_{\pm}$$

holds (cf. §1 for the notation $W^{\pm}$).

For a set $S \subseteq \mathfrak{P}$, we denote by $\mathcal{N}(S)$ the subset of all nilpotent elements in S.

Lemma 2.1. *We can choose Cartan decompositions and moment maps so that the following conditions hold.*

(1) *ψ is an affine quotient map by $K_{\mathbb{C}}$ onto $\mathfrak{s}'$, i.e., $\mathfrak{s}' \simeq W /\!/ K_{\mathbb{C}}$.*

(2) *φ is an affine quotient map by $K'_{\mathbb{C}}$ onto its closed image.*
(3) *$\varphi(\psi^{-1}(\mathcal{N}(\mathfrak{s}'))) \subset \mathcal{N}(\mathfrak{s})$.*

Proof. To show that $\psi : W \to \mathfrak{s}'$ is an affine quotient map, it suffices to show that ψ induces an algebra isomorphism $\mathbb{C}[\mathfrak{s}'] \simeq \mathbb{C}[W]^{K_{\mathbb{C}}}$, where $\mathbb{C}[W]^{K_{\mathbb{C}}}$ denotes the space of invariants of the polynomial ring $\mathbb{C}[W]$. Clearly it suffices to show that the restrictions $\psi\big|_{W^{\pm}}$ induce algebra isomorphisms $\mathbb{C}[\mathfrak{s}'_{\pm}] \simeq \mathbb{C}[W^{\pm}]^{K^{\pm}_{\mathbb{C}}}$.

Similarly, statement (2) asserts that the induced algebra homomorphism $\varphi^* : \mathbb{C}[\mathfrak{s}] \to \mathbb{C}[W]$ maps $\mathbb{C}[\mathfrak{s}]$ onto the $K'_{\mathbb{C}}$-invariants $\mathbb{C}[W]^{K'_{\mathbb{C}}}$. Note that the closed subvariety $\operatorname{Im}\varphi \subset \mathfrak{s}$ is then defined by the ideal of relations of generators of invariants, or in other words, the kernel of φ^*.

Explicit constructions of moment maps for each case of the three dual pairs in Table 1 are summarized in the Appendix. From the explicit formulas given there, the statements (1) – (3) follow easily from classical invariant theory [21]. □

Using the moment maps, we shall define the theta lift of nilpotent orbits from $\mathcal{N}(\mathfrak{s}')$ to $\mathcal{N}(\mathfrak{s})$. Before that, we need the following well-known results due to Kostant and Howe. See Theorem 2.5.4, Proposition 3.7.3.7, and Theorem 3.8.6.4 in [8], and references cited there.

As usual denote by $\mathcal{H}(K^{\pm}_{\mathbb{C}})$ the space of harmonic polynomials on $W^{\pm}$ (under the action of $K^{\pm}_{\mathbb{C}}$). Similar notations apply throughout this article.

Theorem 2.2. *Let (G, G') be in the stable range (see* Table 1) *with G' the smaller member, and consider the dual pairs $(K^{\pm}, G')$ in $Sp(W^{\pm}_{\mathbb{R}})$. Then with respect to the action of $K^{\pm}_{\mathbb{C}}$, we have a tensor product decomposition of $\mathbb{C}[W^{\pm}]$:*

$$\mathbb{C}[W^{\pm}] \simeq \mathcal{H}(K^{\pm}_{\mathbb{C}}) \otimes \mathbb{C}[W^{\pm}]^{K^{\pm}_{\mathbb{C}}}.$$

Moreover as a $K^{\pm}_{\mathbb{C}} \times K'_{\mathbb{C}}$-module, the harmonics $\mathcal{H}(K^{\pm}_{\mathbb{C}})$ decompose without multiplicity.

Corollary 2.3. *We have*

$$\mathbb{C}[W] \simeq \mathcal{H}(K_{\mathbb{C}}) \otimes \mathbb{C}[W]^{K_{\mathbb{C}}} \simeq \mathcal{H}(K_{\mathbb{C}}) \otimes \mathbb{C}[\mathfrak{s}'].$$

Consequently the moment map $\psi : W \to \mathfrak{s}'$ is flat.

Proof. Note that the induced map ψ^* of the moment map ψ is such that

$$\psi^* : \mathbb{C}[\mathfrak{s}'] \simeq \mathbb{C}[W]^{K_{\mathbb{C}}} \hookrightarrow \mathbb{C}[W].$$

Since $\mathbb{C}[W] \simeq \mathcal{H}(K_{\mathbb{C}}) \otimes \mathbb{C}[\mathfrak{s}']$, it is free over $\mathbb{C}[\mathfrak{s}']$. This implies that ψ is flat. □

We introduce some notations. For any complex reductive Lie group E, let $\operatorname{Irr}(E)$ be the set of equivalence classes of irreducible finite dimensional representations of E. Given a (completely reducible) representation of E on $\mathcal{U}$, let $\operatorname{Irr}(E; \mathcal{U})$ be the subset of $\operatorname{Irr}(E)$ which appear in $\mathcal{U}$.

As noted previously, the space $\mathcal{H}(K_{\mathbb{C}}^{+})$ is multiplicity-free as a representation of $K_{\mathbb{C}}^{+} \times K'_{\mathbb{C}}$. Furthermore the decomposition

$$\mathcal{H}(K_{\mathbb{C}}^{+})\big|_{K_{\mathbb{C}}^{+}\times K'_{\mathbb{C}}} \simeq \sum\nolimits^{\oplus}_{\sigma\in\mathrm{Irr}(K_{\mathbb{C}}^{+};\mathcal{H}^{+}),\tau\in\mathrm{Irr}(K'_{\mathbb{C}};\mathcal{H}^{+})} \sigma\boxtimes\tau \tag{2.1}$$

determines a one-to-one correspondence between $\sigma \in \mathrm{Irr}(K_{\mathbb{C}}^{+};\mathcal{H}^{+})$ and $\tau \in \mathrm{Irr}(K'_{\mathbb{C}};\mathcal{H}^{+})$ (see [6]). Similar statements apply for $\mathcal{H}(K_{\mathbb{C}}^{-})$.

We shall abbreviate $R^{\pm}(K'_{\mathbb{C}}) = \mathrm{Irr}(K'_{\mathbb{C}};\mathcal{H}(K_{\mathbb{C}}^{\pm}))$ in the following. Under the assumption of stable range, one can explicitly check that $\tau^{*} \in R^{+}(K'_{\mathbb{C}})$ if and only if $\tau \in R^{-}(K'_{\mathbb{C}})$, where τ denotes the contragredient representation of $\tau \in \mathrm{Irr}(K')$. So we put

$$R(K'_{\mathbb{C}}) = R^{+}(K'_{\mathbb{C}})^{*} = R^{-}(K'_{\mathbb{C}}). \tag{2.2}$$

To summarize, for each $\tau \in R^{\pm}(K'_{\mathbb{C}})$, there is a unique $\sigma \in \mathrm{Irr}(K_{\mathbb{C}}^{\pm})$ such that

$$\mathrm{Hom}_{K_{\mathbb{C}}^{\pm}\times K'_{\mathbb{C}}}(\sigma\boxtimes\tau, \mathcal{H}(K_{\mathbb{C}}^{\pm})) \neq 0.$$

We denote this σ by $\sigma^{\pm}(\tau)$, specifying the dependency of τ and the sign $\pm$. Then, we can rewrite (2.1) and its sister statement as

$$\mathcal{H}(K_{\mathbb{C}}^{+}) \simeq \sum_{\tau\in R(K'_{\mathbb{C}})}^{\oplus} \sigma^{+}(\tau^{*})\boxtimes\tau^{*}, \quad \text{and} \quad \mathcal{H}(K_{\mathbb{C}}^{-}) \simeq \sum_{\tau\in R(K'_{\mathbb{C}})}^{\oplus} \sigma^{-}(\tau)\boxtimes\tau. \tag{2.3}$$

Remark 2.4. By the theory of highest weight, we may identify $R(K'_{\mathbb{C}})$ with a certain semigroup in the integral weight lattice of $K'_{\mathbb{C}}$. We note that in an explicit realization, the semigroup $R(K'_{\mathbb{C}})$ may be identified with a set of partitions, and it is saturated. The explicit decompositions in Equation (2.3) are also well-known. See [10] for the orthogonal and unitary cases, or [8] in general.

Theorem 2.5. *Take a nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}' \subset \mathcal{N}(\mathfrak{s}')$. Then the scheme theoretic fiber $\psi^{-1}(\overline{\mathbb{O}'}) = W \times_{\mathfrak{s}'} \overline{\mathbb{O}'}$ is a reduced, closed irreducible affine subvariety of W. Moreover, it is the closure of a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit in W.*

Proof. Let us prove that the fiber $W \times_{\mathfrak{s}'} \overline{\mathbb{O}'}$ is reduced. For this, it is enough to show that the fiber of each closed point in $\mathfrak{s}'$ is reduced. Namely, we prove the following lemma, which may be of independent interest.

►**[Extra Remarks]** We will supply here the reason why it is enough to do it locally.

We are to prove that the algebra $\mathcal{A} = \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathbb{C}[\overline{\mathbb{O}'}]$ has no nilpotent element. Let us prove it by contradiction. So, assume that $f \in \mathcal{A}, \neq 0$ is nilpotent, i.e., $f^{n} = 0$ for some $n > 0$. Then for any $x \in \overline{\mathbb{O}'}$ we have $f^{n}\big|_{\psi^{-1}(x)} = 0$, which implies $f\big|_{\psi^{-1}(x)} = 0$ because $\psi^{-1}(x)$ is reduced. Since the defining ideal of $\psi^{-1}(x)$ is $\mathbb{I}_{x} = \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathfrak{m}_{x}$ ($\mathfrak{m}_{x}$

is the maximal ideal in $\mathbb{C}[\mathfrak{s}']$ at x), we have

$$\begin{aligned} f \in \bigcap_{x \in \overline{\mathcal{O}'}} \mathbb{I}_x &= \bigcap_{x \in \overline{\mathcal{O}'}} \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathfrak{m}_x \\ &\overset{(\bigstar)}{=} \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \Big(\bigcap_{x \in \overline{\mathcal{O}'}} \mathfrak{m}_x \Big) \qquad \text{(see below for } (\bigstar)) \\ &= \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathbb{I}(\overline{\mathcal{O}'}), \end{aligned}$$

where $\mathbb{I}(\overline{\mathcal{O}'})$ is the defining ideal of $\overline{\mathcal{O}'}$. This means that $f = 0$ on $\psi^{-1}(\overline{\mathcal{O}'})$, which is a contradiction.

Let us prove the equality $(\bigstar)$. Since the inclusion $\supset$ is obvious, we prove

$$\bigcap_{x \in \overline{\mathcal{O}'}} \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathfrak{m}_x \subset \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \Big(\bigcap_{x \in \overline{\mathcal{O}'}} \mathfrak{m}_x \Big).$$

Note that $\mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathfrak{m}_x \simeq \mathcal{H} \otimes \mathfrak{m}_x$. Fix a basis $\{h_i\}_{i \in I}$ in $\mathcal{H}$, and write $f \in \cap_x \mathcal{H} \otimes \mathfrak{m}_x$ as

$$f = \sum_{i \in I} h_i \otimes a_{i,x} \in \mathcal{H} \otimes \mathfrak{m}_x.$$

But this expression means that $a_{i,x} = a_{i,x'} \in \mathfrak{m}_x \cap \mathfrak{m}_{x'}$ for any $x, x' \in \overline{\mathcal{O}'}$, hence we have

$$f \in \mathcal{H} \otimes \big(\cap_x \mathfrak{m}_x \big) = \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \big(\cap_x \mathfrak{m}_x \big).$$

◀

Lemma 2.6. *Let $x \in \mathfrak{s}'$ be a closed point. Then the scheme theoretic fiber $\psi^{-1}(x) = W \times_{\mathfrak{s}'} \{x\}$ is a closed, reduced and irreducible affine subvariety of W. Moreover, it is the closure of a single $K_{\mathbb{C}}$-orbit.*

Proof. The scheme theoretic fiber is defined by

$$W \times_{\mathfrak{s}'} \{x\} = \operatorname{Spec} \left(\mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathbb{C}_x \right).$$

Here $\mathbb{C}_x = \mathbb{C}[\mathfrak{s}']/\mathfrak{m}_x \simeq \mathbb{C}$, where $\mathfrak{m}_x$ denotes the maximal ideal corresponding to the closed point x. Put $\mathcal{A}_x = \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathbb{C}_x$. We are to show that $\mathcal{A}_x$ is an integral domain.

Let us recall the tensor product decomposition

$$\mathbb{C}[W] = \mathcal{H}(K_{\mathbb{C}}) \otimes \mathbb{C}[\mathfrak{s}'].$$

Thus, if we abbreviate $\mathcal{H} = \mathcal{H}(K_{\mathbb{C}})$, we can identify

$$\mathcal{A}_x = \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathbb{C}_x = \mathcal{H} \otimes \mathbb{C}_x \simeq \mathcal{H}.$$

However, we should be careful, because $\mathcal{H}$ does not have any structure of algebra, but only enjoys a structure of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-module. Since $\mathcal{H}$ is a graded subspace of $\mathbb{C}[W]$, it is naturally graded by the ordinary degree of polynomials. We denote the grading by

$$\mathcal{H} \simeq \bigoplus\nolimits_{k \geq 0} \mathcal{H}_k.$$

Then, $\mathcal{A}_x$ becomes a filtered algebra by putting

$$(\mathcal{A}_x)_i = \sum_{k \le i} \mathcal{H}_k \otimes 1 \subset \mathcal{H} \otimes \mathbb{C}_x = \mathcal{A}_x.$$

Now assume that $\mathcal{A}_x$ is not an integral domain. Then there are non-zero elements $a, b \in \mathcal{A}_x$ such that $a \cdot b = 0$. Let us express a and b as

$$a = \sum_{i=0}^{d} a_i \otimes 1 \in \mathcal{H} \otimes \mathbb{C}_x \; ; \quad b = \sum_{j=0}^{d'} b_j \otimes 1 \in \mathcal{H} \otimes \mathbb{C}_x,$$

with $a_d \neq 0$ and $b_{d'} \neq 0$. Then $ab = 0$ implies

$$(a_d b_{d'}) \otimes 1 \in (\mathcal{A}_x)_{d+d'-1},$$

but this is impossible. To see it, let $J = \mathbb{C}[W]_+^{K_{\mathbb{C}}}$ be the augmentation ideal generated by homogeneous invariants of positive degree. Then $\mathcal{H} \cdot J \subset \mathbb{C}[W]$ is a prime ideal defining the null cone $\mathfrak{N}^+ \times \mathfrak{N}^-$, which is known to be irreducible. Note that $\mathcal{H} \cdot J \simeq \mathcal{H} \otimes J$ and $\mathbb{C}[\mathfrak{N}^+ \times \mathfrak{N}^-] = \mathbb{C}[W]/\mathcal{H}J \simeq \mathcal{H}$. Since $a_d b_{d'} \notin \mathcal{H}J$ by the irreducibility of the null cone, we have

$$a_d \cdot b_{d'} \otimes 1 \notin \sum_{k=0}^{d+d'-1} \mathcal{H}_k \otimes \mathbb{C}_x = (\mathcal{A}_x)_{d+d'-1},$$

which is a contradiction.

Next, we prove that the fiber $\psi^{-1}(x)$ contains an open dense $K_{\mathbb{C}}$-orbit. Put $M = \psi^{-1}(x)$ and denote by $\widehat{M}$ the asymptotic cone of M (see [18, §5.2] for the definition of asymptotic cone). Then, by the flatness of ψ, the asymptotic cone $\widehat{M}$ coincides with the null cone $\mathfrak{N}$. Let $\mathcal{O}$ be a generic $K_{\mathbb{C}}$-orbit in M. Consider the cone $\mathbb{C}M$ generated by M in W, then it is clear that the dimension of a generic orbit in $\mathbb{C}M$ is equal to $\dim \mathcal{O}$, which in turn coincides with the dimension of the generic orbit in $\overline{\mathbb{C}M} \subset W$. Since $\mathfrak{N} = \widehat{M} \subset \overline{\mathbb{C}M}$, the dimension of a generic orbit in $\mathfrak{N}$ cannot exceed that of $\mathcal{O}$. Note that $\mathfrak{N}$ has an open dense orbit (this can be verified by a case-by-case analysis, see the Appendix). This means that $\dim \mathcal{O} \ge \dim \mathfrak{N}$. On the other hand, we have an equality $\dim M = \dim \widehat{M} = \dim \mathfrak{N}$ of dimensions, hence $\dim \mathcal{O} \ge \dim M$. Since $\mathcal{O} \subset M$, we conclude that $\dim \mathcal{O} = \dim M$, and that $\mathcal{O}$ is an open dense orbit in M, by the irreducibility of M just proved above. □

Let us return to the proof of Theorem 2.5.

By $K'_{\mathbb{C}}$-equivariancy of ψ, we get $\psi^{-1}(\mathbb{O}') = K'_{\mathbb{C}} \cdot \psi^{-1}(\{x\})$ for any $x \in \mathbb{O}'$. Consider the multiplication map $K'_{\mathbb{C}} \times \psi^{-1}(\{x\}) \to \psi^{-1}(\mathbb{O}')$. Since $K'_{\mathbb{C}} \times \psi^{-1}(\{x\})$ is irreducible by the above lemma, $\psi^{-1}(\mathbb{O}')$ is also irreducible (as an image of an irreducible set). Therefore its closure $\overline{\psi^{-1}(\mathbb{O}')}$ is irreducible. Since the moment map ψ is flat by Corollary 2.3, it is an open map ([4, Ex. (III.9.1)]). Thus we conclude that $\psi^{-1}(\overline{\mathbb{O}'}) = \overline{\psi^{-1}(\mathbb{O}')}$ is irreducible.

By the same lemma, $\psi^{-1}(x)$ contains an open dense $K_{\mathbb{C}}$-orbit $\mathcal{O}_x$. Then the union of $K'_{\mathbb{C}}$-translates $K'_{\mathbb{C}} \cdot \mathcal{O}_x$ is dense in $\psi^{-1}(\mathbb{O}')$. Thus $\overline{K'_{\mathbb{C}} \cdot \mathcal{O}_x} = \overline{\psi^{-1}(\mathbb{O}')} = \psi^{-1}(\overline{\mathbb{O}'})$, which means that $K'_{\mathbb{C}} \cdot \mathcal{O}_x$ is open in $\psi^{-1}(\overline{\mathbb{O}'})$. □

Remark 2.7. Recently, T. Ohta proved the irreducibility and the existence of an open dense orbit by a totally different (but case-by-case) method ([16], [17]). In fact, his method is applicable beyond the stable range. However, outside the stable range, things get much more complicated and the same statement of the above theorem is no longer true.

From the above theorem, we see that $\varphi(\psi^{-1}(\overline{\mathbb{O}'}))$ is a $K_{\mathbb{C}}$-stable irreducible closed set in $\mathfrak{s}$. Since $\varphi(\psi^{-1}(\overline{\mathbb{O}'}))$ is contained in the nilpotent variety $\mathcal{N}(\mathfrak{s})$, which has only finite number of $K_{\mathbb{C}}$-orbits, it must be the closure of a single $K_{\mathbb{C}}$-orbit $\mathbb{O}$.

Definition 2.8. Let $\mathbb{O}' \subset \mathfrak{s}'$ be a nilpotent $K'_{\mathbb{C}}$-orbit. Then $\psi^{-1}(\overline{\mathbb{O}'})$ is a closed, irreducible affine subvariety in W. The image $\varphi(\psi^{-1}(\overline{\mathbb{O}'}))$ is the closure of a single nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{s}$, i.e., $\varphi(\psi^{-1}(\overline{\mathbb{O}'})) = \overline{\mathbb{O}}$ for a certain nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}$. We call $\mathbb{O}$ the *theta lift* of $\mathbb{O}'$, and denote it by $\theta(\mathbb{O}')$.

Recall the subset $R(K'_{\mathbb{C}}) \subset \mathrm{Irr}(K'_{\mathbb{C}})$ defined in Equation (2.2).

Theorem 2.9. *Let (G, G') be in the stable range (see* Table 1*) with G' the smaller member. For a nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}' \subset \mathcal{N}(\mathfrak{s}')$, denote its theta lift by $\mathbb{O} = \theta(\mathbb{O}')$. Put $\Xi(\mathbb{O}') = \psi^{-1}(\overline{\mathbb{O}'})$. Then the closure $\overline{\mathbb{O}}$ is an affine quotient of $\Xi(\mathbb{O}')$ by $K'_{\mathbb{C}}$.*

$$\overline{\mathbb{O}} \simeq \Xi(\mathbb{O}')//K'_{\mathbb{C}}.$$

Moreover, the $K_{\mathbb{C}}$-module structure of the regular function ring of $\overline{\mathbb{O}}$ is given in terms of $\overline{\mathbb{O}'}$ as follows:

$$\begin{aligned} \mathbb{C}[\overline{\mathbb{O}}] &\simeq \big(\mathcal{H}(K_{\mathbb{C}}) \otimes \mathbb{C}[\overline{\mathbb{O}'}]\big)^{K'_{\mathbb{C}}} \\ &\simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \mathrm{Hom}_{K'_{\mathbb{C}}}(\tau_1 \otimes \tau_2^*, \mathbb{C}[\overline{\mathbb{O}'}]) \otimes \big(\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big), \end{aligned} \tag{2.4}$$

where $\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)$ is an irreducible representation of $K_{\mathbb{C}} = K_{\mathbb{C}}^+ \times K_{\mathbb{C}}^-$ given in Equation (2.3)*, and $K_{\mathbb{C}}$ acts on the space of multiplicities* $\mathrm{Hom}_{K'_{\mathbb{C}}}(\tau_1 \otimes \tau_2^*, \mathbb{C}[\overline{\mathbb{O}'}])$ *trivially.*

Proof. The assertion that $\overline{\mathbb{O}}$ is an affine quotient follows from general theory of affine quotient maps because φ itself is a quotient map, and $\Xi(\mathbb{O}')$ is a $K'_{\mathbb{C}}$-invariant, affine closed subvariety of W (cf. [13, Prop. 3.3]).

Next we prove the statement on the module structure of $\mathbb{C}[\overline{\mathbb{O}}]$. By Theorem 2.5, we have

$$\begin{aligned}
\mathbb{C}[\Xi(\mathbb{O}')] &\simeq \mathbb{C}[W \times_{\mathfrak{s}'} \overline{\mathbb{O}'}] = \mathbb{C}[W] \otimes_{\mathbb{C}[\mathfrak{s}']} \mathbb{C}[\overline{\mathbb{O}'}] \\
&\simeq \big(\mathbb{C}[W^+] \otimes \mathbb{C}[W^-]\big) \otimes_{\mathbb{C}[\mathfrak{s}'_+]\otimes\mathbb{C}[\mathfrak{s}'_-]} \mathbb{C}[\overline{\mathbb{O}'}] \\
&\simeq \Big(\mathbb{C}[W^+] \otimes_{\mathbb{C}[\mathfrak{s}'_+]} \mathbb{C}[\overline{\mathbb{O}'}]\Big) \otimes_{\mathbb{C}[\overline{\mathbb{O}'}]} \Big(\mathbb{C}[W^-] \otimes_{\mathbb{C}[\mathfrak{s}'_-]} \mathbb{C}[\overline{\mathbb{O}'}]\Big). \qquad (2.5)
\end{aligned}$$

Note that $\mathbb{C}[W^\pm] \simeq \mathcal{H}(K_\mathbb{C}^\pm)\otimes\mathbb{C}[W^\pm]^{K_\mathbb{C}^\pm}$ and $\mathbb{C}[W^\pm]^{K_\mathbb{C}^\pm} \simeq \mathbb{C}[\mathfrak{s}'_\pm]$ by Theorem 2.2. Therefore we have

$$\mathbb{C}[W^+] \otimes_{\mathbb{C}[\mathfrak{s}'_+]} \mathbb{C}[\overline{\mathbb{O}'}] \simeq \big(\mathcal{H}(K_\mathbb{C}^+) \otimes \mathbb{C}[W^+]^{K_\mathbb{C}^+}\big) \otimes_{\mathbb{C}[\mathfrak{s}'_+]} \mathbb{C}[\overline{\mathbb{O}'}] \simeq \mathcal{H}(K_\mathbb{C}^+) \otimes \mathbb{C}[\overline{\mathbb{O}'}],$$

and (2.5) becomes

$$\begin{aligned}
(2.5) \ &= \big(\mathcal{H}(K_\mathbb{C}^+) \otimes \mathbb{C}[\overline{\mathbb{O}'}]\big) \otimes_{\mathbb{C}[\overline{\mathbb{O}'}]} \big(\mathcal{H}(K_\mathbb{C}^-) \otimes \mathbb{C}[\overline{\mathbb{O}'}]\big) \\
&\simeq \big(\mathcal{H}(K_\mathbb{C}^+) \otimes \mathcal{H}(K_\mathbb{C}^-)\big) \otimes \mathbb{C}[\overline{\mathbb{O}'}].
\end{aligned}$$

Since φ is an affine quotient map by the action of $K'_\mathbb{C}$, we have

$$\begin{aligned}
\mathbb{C}[\overline{\mathbb{O}}] &\simeq \mathbb{C}[\Xi(\mathbb{O}')]^{K'_\mathbb{C}} \simeq \Big(\big(\mathcal{H}(K_\mathbb{C}^+) \otimes \mathcal{H}(K_\mathbb{C}^-)\big) \otimes \mathbb{C}[\overline{\mathbb{O}'}]\Big)^{K'_\mathbb{C}} \\
&\simeq \sum_{\tau_1,\tau_2\in R(K'_\mathbb{C})}^{\oplus} \Big(\big(\sigma^+(\tau_1^*) \boxtimes \tau_1^*\big) \otimes \big(\sigma^-(\tau_2) \boxtimes \tau_2\big) \otimes \mathbb{C}[\overline{\mathbb{O}'}]\Big)^{K'_\mathbb{C}} \\
&\simeq \sum_{\tau_1,\tau_2\in R(K'_\mathbb{C})}^{\oplus} \big(\tau_1^* \otimes \tau_2 \otimes \mathbb{C}[\overline{\mathbb{O}'}]\big)^{K'_\mathbb{C}} \otimes \big(\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big) \\
&\simeq \sum_{\tau_1,\tau_2\in R(K'_\mathbb{C})}^{\oplus} \operatorname{Hom}_{K'_\mathbb{C}}(\tau_1 \otimes \tau_2^*, \mathbb{C}[\overline{\mathbb{O}'}]) \otimes \big(\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big).
\end{aligned}$$

The theorem follows. $\square$

3. Lifting of holomorphic nilpotent orbits

We apply Theorem 2.9 to the lifting of the so-called holomorphic nilpotent orbits. In the process, we get a family of spherical orbits and prove normality of the closure of certain lifted orbits. This application reproduces some of our previous results ([12], [13], [15]) as well. In the following, we use the notations in Theorem 2.9 freely.

Let us consider the orbit decomposition of $\mathfrak{s}'_+$ by the adjoint action of $K'_\mathbb{C}$. Any $K'_\mathbb{C}$-orbit in $\mathfrak{s}'_+$ is clearly nilpotent. We call these nilpotent orbit *holomorphic*. It is well known that $\mathfrak{s}'_+$ is a prehomogeneous vector space, and there exists a numbering of $K'_\mathbb{C}$-orbits $\mathbb{O}'_0, \mathbb{O}'_1, \dots, \mathbb{O}'_l$ in such a way that $\mathbb{O}'_{i-1} \subset \overline{\mathbb{O}'_i}$ for $1 \le i \le l$. Here l is the real rank of G'. As a consequence, $\mathbb{O}'_0 = \{0\}$ and $\overline{\mathbb{O}'_l} = \mathfrak{s}'_+$, i.e., $\mathbb{O}'_l$ is the open dense orbit in $\mathfrak{s}'_+$. The orbit $\mathbb{O}'_l$ is called *regular*, while the orbits $\{\mathbb{O}'_k\}_{0\le k<l}$ are called *singular*.

3.1. **Trivial orbit.** We first consider the trivial orbit $\mathbb{O}'^{\mathbf{1}} = \mathbb{O}'_0 = \{0\}$. Then, the lifted orbit $\mathbb{O}^{\mathbf{1}} = \theta(\mathbb{O}'^{\mathbf{1}})$ is a two-step nilpotent orbit in $\mathcal{N}(\mathfrak{s})$ (cf. [12]). We have

Theorem 3.1. *The group $K_{\mathbb{C}}$ acts on $\overline{\mathbb{O}^{\mathbf{1}}}$ multiplicity-freely. As a $K_{\mathbb{C}} = K_{\mathbb{C}}^+ \times K_{\mathbb{C}}^-$-module, we have*

$$\mathbb{C}[\overline{\mathbb{O}^{\mathbf{1}}}] \simeq \sum_{\tau \in R(K'_{\mathbb{C}})}^{\oplus} \sigma^+(\tau^*) \boxtimes \sigma^-(\tau). \tag{3.1}$$

Proof. Since $\mathbb{C}[\mathbb{O}'^{\mathbf{1}}] = \mathbb{C}$, by Theorem 2.9, we get

$$\mathbb{C}[\overline{\mathbb{O}^{\mathbf{1}}}] \simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \mathrm{Hom}_{K'_{\mathbb{C}}}(\tau_1 \otimes \tau_2^*, \mathbb{C}) \otimes \big(\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big).$$

But, by Schur's lemma, the multiplicity $\mathrm{Hom}_{K'_{\mathbb{C}}}(\tau_1 \otimes \tau_2^*, \mathbb{C})$ is not zero if and only if $\tau_1 = \tau_2$, in which case it is $\mathbb{C}$, i.e., the multiplicity is one. □

Recall that an orbit $\mathbb{O}$ is called *spherical*, or more precisely $K_{\mathbb{C}}$*-spherical*, if a Borel subgroup of $K_{\mathbb{C}}$ has a dense orbit in $\mathbb{O}$. This is equivalent to saying that $K_{\mathbb{C}}$ acts on $\overline{\mathbb{O}}$ multiplicity-freely (i.e., $\mathbb{C}[\overline{\mathbb{O}}]$ decomposes without multiplicity as a $K_{\mathbb{C}}$-module).

Corollary 3.2. *The orbit $\mathbb{O}^{\mathbf{1}}$ lifted from the trivial orbit is spherical. The closure $\overline{\mathbb{O}^{\mathbf{1}}}$ is a normal variety.*

Proof. The claim that $\mathbb{O}^{\mathbf{1}}$ is spherical follows directly from the above theorem. The normality follows from [20, Th. 10]. □

3.2. **Singular holomorphic orbits.** In this subsection, we are concerned with the case of singular holomorphic orbits, although the results also hold for the regular holomorphic orbit. We have more to say in the next subsection on the latter case.

Consider the orbit $\mathbb{O}'_k$, where $0 \leq k \leq l$. In the following, we recall the geometric structure of $\mathbb{O}'_k$, and the decomposition of the regular function ring $\mathbb{C}[\overline{\mathbb{O}'_k}]$ by the action of $K'_{\mathbb{C}}$. For details, see §7 of [14].

There exists a symplectic space $W(k)_{\mathbb{R}}$ such that $(G(k), G')$ forms a compact dual pair in $Sp(W(k)_{\mathbb{R}})$. There is also a complex structure on $W(k)_{\mathbb{R}}$, such that, if we regard $W(k) = W(k)_{\mathbb{R}}$ as a complex vector space with respect to the complex structure, then the imaginary part of the Hermitian form provides the original symplectic form on $W(k)_{\mathbb{R}}$. The complex space $W(k)$ carries a natural linear action of $G(k)_{\mathbb{C}} \times K'_{\mathbb{C}}$, and we have $\overline{\mathbb{O}'_k} = W(k)//G(k)_{\mathbb{C}}$, an affine quotient of $W(k)$ by the action of $G(k)_{\mathbb{C}}$. Moreover, there is a larger compact group $L(k)$ which contains $G(k)$ as a symmetric subgroup, and the pair $(L(k), K')$ forms a dual pair in $Sp(W(k)_{\mathbb{R}})$ (cf. §1). From the standard result of Howe [6], we know that $L(k)_{\mathbb{C}} \times K'_{\mathbb{C}}$ acts on $W(k)$ in a multiplicity-free manner:

$$\mathbb{C}[W(k)] \simeq \sum_{\tau \in \mathrm{Irr}(K'_{\mathbb{C}}; W(k))}^{\oplus} \rho_k(\tau) \boxtimes \tau, \tag{3.2}$$

where $\rho_k(\tau) \in \mathrm{Irr}(L(k)_{\mathbb{C}}; W(k))$ corresponds to $\tau \in \mathrm{Irr}(K'_{\mathbb{C}}; W(k))$ via the above multiplicity-free action.

The following theorem is proved in [14].

Theorem 3.3. *$\mathbb{O}'_k$ is a $K'_{\mathbb{C}}$-spherical nilpotent orbit, and we have the following multiplicity-free decomposition:*

$$\mathbb{C}[\overline{\mathbb{O}'_k}] \simeq \sum_{\tau \in \mathrm{Irr}(K'_{\mathbb{C}}; \mathbb{O}'_k)}^{\oplus} \tau,$$

where

$$\mathrm{Irr}(K'_{\mathbb{C}}; \mathbb{O}'_k) = \{\tau \in \mathrm{Irr}(K'_{\mathbb{C}}; W(k)) \mid \ \rho_k(\tau) \text{ has a non-zero } G(k)_{\mathbb{C}}\text{-fixed vector}\}.$$

Now let us denote $\mathbb{O}^{\mathrm{hol}}_k = \theta(\mathbb{O}'_k)$, the theta lift of the holomorphic orbit $\mathbb{O}'_k$. Then $\mathbb{O}^{\mathrm{hol}}_k$ is a three-step nilpotent orbit in $\mathcal{N}(\mathfrak{s})$ (cf. [13]).

As an immediate consequence of Theorems 2.9 and 3.3, we have

Theorem 3.4. *For $\tau, \tau_1, \tau_2 \in \mathrm{Irr}(K'_{\mathbb{C}})$, let us denote by $m(\tau_1, \tau_2 \otimes \tau)$ the multiplicity of τ_1 in the tensor product representation $\tau_2 \otimes \tau$. Then we have the following decomposition.*

$$\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}] \simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \Big(\sum_{\tau \in \mathrm{Irr}(K'_{\mathbb{C}}; \mathbb{O}'_k)} m(\tau_1, \tau_2 \otimes \tau) \Big) \, \sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2), \tag{3.3}$$

where $R(K'_{\mathbb{C}}) = \mathrm{Irr}(K'_{\mathbb{C}}; \mathcal{H}(K^-_{\mathbb{C}}))$, as before.

3.3. **Regular holomorphic orbit.** Now consider $\mathbb{O}^{\mathrm{hol}}_l$ lifted from the regular holomorphic orbit $\mathbb{O}'_l \subset \mathfrak{s}'_+$. This example is already covered in the previous subsection. However the orbit $\mathbb{O}^{\mathrm{hol}}_l$ has a new property which is not shared by the lifts of singular holomorphic orbits. This property allows us to conclude the normality of $\overline{\mathbb{O}^{\mathrm{hol}}_l}$ as well as to have a more convenient description of $\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_l}]$.

For the regular orbit, we have

$$\Xi(\mathbb{O}'_l) = \psi^{-1}(\overline{\mathbb{O}'_l}) = \psi^{-1}(\mathfrak{s}'_+) = W^+ \times \mathfrak{N}^-,$$

where $\psi^- = \psi\big|_{W^-}$ and $\mathfrak{N}^- = (\psi^-)^{-1}(0) \subset W^-$ is a null cone. Thus Theorem 2.9 yields the following

Lemma 3.5. *Let $\mathbb{O}^{\mathrm{hol}}_l \subset \mathcal{N}(\mathfrak{s})$ be the theta lift of the regular holomorphic orbit $\mathbb{O}'_l$, which is open dense in $\mathfrak{s}'_+$. Then its closure is an affine quotient of $W^+ \times \mathfrak{N}^-$ by $K'_{\mathbb{C}}$,*

$$\overline{\mathbb{O}^{\mathrm{hol}}_l} \simeq W^+ \times \mathfrak{N}^- // K'_{\mathbb{C}}.$$

►[**Extra Remarks**] On the claim that (vector space) × (normal variety) is again normal:

This can be proved as follows (I have learned this from Makoto Matumoto). What should be proved is the following algebraic statement.

Proposition 3.A. *Let A be an integrally closed domain over $\mathbb{C}$. Then the polynomial algebra $A[x_1, \ldots, x_n]$ over A is integrally closed.*

To prove this, it is enough to assume that $n = 1$. The following lemmas are well known.

Lemma 3.B. *If A is a discrete valuation ring, then $A[x]$ is a UFD. In particular, $A[x]$ is integrally closed.*

Lemma 3.C. *If A is an integrally closed domain, then $A = \bigcap_{\mathfrak{p}\text{: height } 1} A_{\mathfrak{p}}$.*

Let $K = Q(A)$ be the quotient field of A. By the first lemma, we know that $A_{\mathfrak{p}}[x]$ is integrally closed in $Q(A_{\mathfrak{p}})(x)$. Since the integral closure of $A[x]$ in $K(x)$ is contained in the integral closure of $A_{\mathfrak{p}}[x]$ in $K(x) = Q(A_{\mathfrak{p}})(x)$ for any $\mathfrak{p}$, we have

$$\text{int. closure}(A[x]) \subset \bigcap_{\mathfrak{p}\text{: height } 1} A_{\mathfrak{p}}[x] = A[x].$$

This completes the proof of the proposition. ◀

Let us recall the compact group L^+ whose action on W^+ commutes with that of K'. In fact we have the dual pair $(L^+, K') \subset Sp(W^+_{\mathbb{R}})$. Let

$$\mathbb{C}[W^+] \simeq \sum_{\tau \in \mathrm{Irr}(K'_{\mathbb{C}}; \mathbb{C}[W^+])}^{\oplus} \rho^+(\tau) \otimes \tau \tag{3.4}$$

be the decomposition of $\mathbb{C}[W^+]$ as a $L^+ \times K'$-module. Here $\rho^+(\tau) \in \mathrm{Irr}(L^+)$ corresponds to $\tau \in \mathrm{Irr}(K')$ via the above multiplicity-free decomposition. Note that $\mathbb{C}[W^+] \simeq \mathcal{H}(K^+_{\mathbb{C}}) \otimes \mathbb{C}[W^+]^{K^+_{\mathbb{C}}}$, and so $\mathrm{Irr}(K'_{\mathbb{C}}; \mathbb{C}[W^+]) = \mathrm{Irr}(K'_{\mathbb{C}}; \mathcal{H}(K^+_{\mathbb{C}})) = R(K'_{\mathbb{C}})^*$. We may therefore rewrite Equation (3.4) as

$$\mathbb{C}[W^+] \simeq \sum_{\tau \in R(K'_{\mathbb{C}})}^{\oplus} \rho^+(\tau^*) \otimes \tau^*. \tag{3.5}$$

Note that L^+ is a unitary group containing K^+ in each of the three cases, hence its complexification $L^+_{\mathbb{C}}$ is isomorphic to a general linear group. Since the action of $L^+_{\mathbb{C}}$ commutes with that of $K'_{\mathbb{C}}$, $L^+_{\mathbb{C}} \times K^-_{\mathbb{C}}$ naturally acts on the affine quotient space

$$W^+ \times \mathfrak{N}^- // K'_{\mathbb{C}} \simeq \overline{\mathbb{O}^{\mathrm{hol}}_l},$$

extending the action of $K_{\mathbb{C}} = K^+_{\mathbb{C}} \times K^-_{\mathbb{C}}$.

Remark 3.6. The orbit $\mathbb{O}^{\mathrm{hol}}_l$ itself does *not* admit an action of $L^+_{\mathbb{C}} \times K^-_{\mathbb{C}}$, but its closure does.

Theorem 3.7. *The group $L^+_{\mathbb{C}} \times K^-_{\mathbb{C}}$ acts on the closure $\overline{\mathbb{O}^{\mathrm{hol}}_l}$ multiplicity-freely. As a $L^+_{\mathbb{C}} \times K^-_{\mathbb{C}}$-module, we have*

$$\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_l}] \simeq \sum_{\tau \in R(K'_{\mathbb{C}})}^{\oplus} \rho^+(\tau^*) \boxtimes \sigma^-(\tau). \tag{3.6}$$

Proof. We have

$$\begin{aligned}
\mathbb{C}[\overline{\mathbb{O}_l^{\mathrm{hol}}}] &\simeq \big(\mathbb{C}[W^+] \otimes \mathbb{C}[\mathfrak{N}^-]\big)^{K'_{\mathbb{C}}} \\
&\simeq \Big(\Big(\sum_{\tau_1 \in R(K'_{\mathbb{C}})}^{\oplus} \rho^+(\tau_1^*) \otimes \tau_1^*\Big) \otimes \Big(\sum_{\tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \sigma^-(\tau_2) \otimes \tau_2\Big)\Big)^{K'_{\mathbb{C}}} \\
&\simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} (\tau_1^* \otimes \tau_2)^{K'_{\mathbb{C}}} \otimes \big(\rho^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big) \\
&\simeq \sum_{\tau \in R(K'_{\mathbb{C}})}^{\oplus} \rho^+(\tau^*) \boxtimes \sigma^-(\tau).
\end{aligned}$$

□

Corollary 3.8. *The orbit $\mathbb{O}_l^{\mathrm{hol}}$ is $L_{\mathbb{C}}^+ \times K_{\mathbb{C}}^-$-spherical. The closure $\overline{\mathbb{O}_l^{\mathrm{hol}}}$ is a normal variety.*

Proof. The first statement follows directly from the above theorem. The normality follows from [20, Th. 10], since $\mathbb{Z}R(K'_{\mathbb{C}}) \cap \mathbb{Q}_+ R(K'_{\mathbb{C}}) = R(K'_{\mathbb{C}})$. □

Next we consider the $K_{\mathbb{C}}$-module structure of the regular function ring on $\overline{\mathbb{O}_l^{\mathrm{hol}}}$. Since $\overline{\mathbb{O}'_l} = \mathfrak{s}'_+$, Theorem 2.9 yields

$$\begin{aligned}
\mathbb{C}[\overline{\mathbb{O}_l^{\mathrm{hol}}}] &\simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \mathrm{Hom}_{K'_{\mathbb{C}}}(\tau_1, \tau_2 \otimes \mathbb{C}[\mathfrak{s}'_+]) \otimes \big(\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big) \\
&\simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \mathrm{Hom}_{K'_{\mathbb{C}}}(\tau_2, \tau_1 \otimes \mathbb{C}[\mathfrak{s}'_-]) \otimes \big(\sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2)\big).
\end{aligned}$$

For any $\tau_1, \tau_2 \in \mathrm{Irr}(K')$, we define the branching coefficient $b_{\tau_2}^{\tau_1}$ by the following formula:

$$\tau_1 \otimes \mathbb{C}[\mathfrak{s}'_-] \simeq \sum_{\tau_2 \in \mathrm{Irr}(K')}^{\oplus} b_{\tau_2}^{\tau_1}\, \tau_2. \tag{3.7}$$

Thus we get

$$\mathbb{C}[\overline{\mathbb{O}_l^{\mathrm{hol}}}] \simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} b_{\tau_2}^{\tau_1}\, \sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2). \tag{3.8}$$

Remark 3.9. Let us denote by $D(\tau)$ a holomorphic discrete series representation of G' with the extreme K'-type τ. Then it is well known that

$$D(\tau)\big|_{K'} \simeq \tau \otimes \mathbb{C}[\mathfrak{s}'_-]$$

as K'-modules. For any $\tau_1 \in \mathrm{Irr}(K')$, there exists a suitable unitary character χ of K' such that $\tau = \tau_1 \otimes \chi$ is the extreme K'-type of a holomorphic discrete series representation.

Then our branching coefficient $b^{\tau_1}_{\tau_2}$ satisfies

$$D(\tau_1 \otimes \chi)\big|_{K'} \simeq \sum_{\tau_2 \in \mathrm{Irr}(K')}^{\oplus} b^{\tau_1}_{\tau_2} \, (\tau_2 \otimes \chi),$$

and it does not depend on the choice of the unitary character χ. Hence, Equation (3.8) tells us that the orbit structure of $\mathbb{O}^{\mathrm{hol}}_l$ encodes the K'-type decomposition of (all) holomorphic discrete series of G'.

We shall show that the branching coefficient $b^{\tau_1}_{\tau_2}$ may be expressed in terms of the branching coefficient of finite dimensional representations.

For $\tau_1, \tau_2 \in R(K'_{\mathbb{C}})$, define the branching coefficient $c^{\tau_2}_{\tau_1}$ by the following restriction formula:

$$\rho^+(\tau_2^*)\big|_{K^+_{\mathbb{C}}} \simeq \sum_{\tau_1 \in R(K'_{\mathbb{C}})}^{\oplus} c^{\tau_2}_{\tau_1} \, \sigma^+(\tau_1^*). \tag{3.9}$$

Theorem 3.10. *If we define the branching coefficient $b^{\tau_1}_{\tau_2}$ and $c^{\tau_2}_{\tau_1}$ by (3.7) and (3.9) respectively, then*

$$b^{\tau_1}_{\tau_2} = c^{\tau_2}_{\tau_1}, \qquad \forall \tau_1, \tau_2 \in R(K'_{\mathbb{C}}).$$

Consequently we have

$$\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_l}] \simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} b^{\tau_1}_{\tau_2} \, \sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2) \simeq \sum_{\tau_1, \tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} c^{\tau_2}_{\tau_1} \, \sigma^+(\tau_1^*) \boxtimes \sigma^-(\tau_2). \tag{3.10}$$

Proof. From Equation (3.5), we have

$$\begin{aligned} \mathbb{C}[W^+] &\simeq \sum_{\tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} \rho^+(\tau_2^*) \otimes \tau_2^* \\ &\simeq \sum_{\tau_1 \in R(K'_{\mathbb{C}})}^{\oplus} \sigma^+(\tau_1^*) \boxtimes \Big(\sum_{\tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} c^{\tau_2}_{\tau_1} \, \tau_2^* \Big) \qquad (\text{as } K^+_{\mathbb{C}} \times K'_{\mathbb{C}}\text{-module}). \end{aligned}$$

Comparing this with

$$\mathbb{C}[W^+] \simeq \mathcal{H}(K^+_{\mathbb{C}}) \otimes \mathbb{C}[\mathfrak{s}'_+] \simeq \sum_{\tau_1 \in R(K'_{\mathbb{C}})}^{\oplus} \sigma^+(\tau_1^*) \otimes \big(\tau_1 \otimes \mathbb{C}[\mathfrak{s}'_-]\big)^*,$$

we get

$$\tau_1 \otimes \mathbb{C}[\mathfrak{s}'_-] \simeq \sum_{\tau_2 \in R(K'_{\mathbb{C}})}^{\oplus} c^{\tau_2}_{\tau_1} \, \tau_2, \qquad \text{i.e.,} \qquad b^{\tau_1}_{\tau_2} = c^{\tau_2}_{\tau_1}.$$

$\square$

4. Appendix

In Appendix, we will give an explicit construction of moment maps φ and ψ, and establish some fundamental properties of null cones and harmonics.

4.1. $O(p,q) \times Sp(2n,\mathbb{R})$ $(2n < \min(p,q))$. This case is already treated in [12]. For convenience of the readers, we reproduce the synopsis of the arguments, and at the same time we improve some of statements there.

Put $G = O(p,q)$ and $G' = Sp(2n,\mathbb{R})$, and assume the stable range condition $2n < p,q$. We denote:

$$\begin{array}{ccc} M = U(p,q) & \supset & G = O(p,q) \\ \cup & & \cup \\ L = L^+ \times L^- = U(p) \times U(q) & \supset & K = K^+ \times K^- = O(p) \times O(q) \end{array}$$

$$\begin{array}{ccc} M' = Sp(2n,\mathbb{R})^2 & \overset{\Delta}{\supset} & G' = Sp(2n,\mathbb{R}) \\ \cup & & \cup \\ L' = K' \times K' = U(n)^2 & \overset{\Delta}{\supset} & K' = U(n) \end{array}$$

In this diagram, the vertical containment means respectively maximal compact subgroups, and Δ denotes the diagonal embedding. Put

$$\begin{aligned} W = M_{p+q,n}(\mathbb{C}) &= \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \mid A \in M_{p,n}(\mathbb{C}), B \in M_{q,n}(\mathbb{C}) \right\} \\ &= W^+ \oplus W^- = M_{p,n}(\mathbb{C}) \oplus M_{q,n}(\mathbb{C}). \end{aligned}$$

The complexification $K_{\mathbb{C}} = O(p,\mathbb{C}) \times O(q,\mathbb{C})$ acts on W via the left multiplication, and $K'_{\mathbb{C}} = GL_n(\mathbb{C})$ acts on W as

$$g \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A\,{}^t g \\ B g^{-1} \end{pmatrix}, \qquad g \in GL_n(\mathbb{C}).$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ (resp. $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$) as

$$\mathfrak{g} = \mathfrak{o}(p+q,\mathbb{C}) = \left(\begin{array}{c|c} \mathrm{Alt}_p(\mathbb{C}) & 0 \\ \hline 0 & \mathrm{Alt}_q(\mathbb{C}) \end{array} \right) \oplus \left(\begin{array}{c|c} 0 & M_{p,q}(\mathbb{C}) \\ \hline {}^t M_{p,q}(\mathbb{C}) & 0 \end{array} \right) = \mathfrak{k} \oplus \mathfrak{s},$$

$$\mathfrak{g}' = \mathfrak{sp}(2n,\mathbb{C}) = \left(\begin{array}{c|c} M_n(\mathbb{C}) & 0 \\ \hline 0 & -{}^t M_n(\mathbb{C}) \end{array} \right) \oplus \left(\begin{array}{c|c} 0 & \mathrm{Sym}_n(\mathbb{C}) \\ \hline \mathrm{Sym}_n(\mathbb{C}) & 0 \end{array} \right) = \mathfrak{k}' \oplus \mathfrak{s}'.$$

Thus, we can identify $\mathfrak{s} = M_{p,q}(\mathbb{C})$ and $\mathfrak{s}' = \mathfrak{s}'_+ \oplus \mathfrak{s}'_- = \mathrm{Sym}\,_n(\mathbb{C}) \oplus \mathrm{Sym}\,_n(\mathbb{C})$. The moment maps are explicitly given by

$$\varphi : W \ni \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto A\,{}^tB \in \mathfrak{s}, \qquad \psi : W \ni \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto ({}^tAA, {}^tBB) \in \mathfrak{s}'.$$

These maps are clearly $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant if we define the trivial $K_{\mathbb{C}}$-action (resp. $K'_{\mathbb{C}}$-action) on $\mathfrak{s}'$ (resp. $\mathfrak{s}$). In the language of algebra, φ is induced by

$$\varphi^*(z_{i,j}) = \sum_{k=1}^{n} a_{i,k} b_{j,k}, \qquad Z = (z_{i,j}) \in \mathfrak{s}, \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} (a_{i,j}) \\ (b_{i,j}) \end{pmatrix} \in W,$$

and it is well known that $\varphi^*(z_{i,j})$'s generate GL_n-invariants $\mathbb{C}[W]^{K'_{\mathbb{C}}}$. This shows that $\varphi : W \to \mathfrak{s}$ is an affine quotient map onto its closed image:

$$\mathfrak{s} \supset \varphi(W) \simeq W/\!/K'_{\mathbb{C}}.$$

Similarly, ψ is induced by

$$\psi^*(x_{i,j}) = \sum_{k=1}^{p} a_{k,i} a_{k,j}, \quad X = (x_{i,j}) \in \mathfrak{s}'_+, \qquad \psi^*(y_{i,j}) = \sum_{l=1}^{q} b_{l,i} b_{l,j}, \quad Y = (y_{i,j}) \in \mathfrak{s}'_-.$$

Again, $\psi(x_{i,j})^*$'s and $\psi(y_{i,j})^*$'s generate the $O(p,\mathbb{C}) \times O(q,\mathbb{C})$-invariants $\mathbb{C}[W]^{K_{\mathbb{C}}}$, which proves that $\psi : W \to \mathfrak{s}'$ is an affine quotient map. By the assumption of the stable range, $2n < p, q$, it is easy to see that $\psi : W \to \mathfrak{s}'$ is a surjection, hence

$$\begin{aligned} &\mathfrak{s}' \simeq W/\!/K_{\mathbb{C}}, \\ &\mathbb{C}[\mathfrak{s}'] = \mathbb{C}[\mathrm{Sym}\,_n] \otimes \mathbb{C}[\mathrm{Sym}\,_n] \simeq \mathbb{C}[M_{p+q,n}]^{O(p,\mathbb{C}) \times O(q,\mathbb{C})} = \mathbb{C}[W]^{K_{\mathbb{C}}}. \end{aligned}$$

Let us consider the null cones

$$\mathfrak{N} = \psi^{-1}(0) \subset W, \qquad \mathfrak{N}^{\pm} = (\psi^{\pm})^{-1}(0) \subset W^{\pm},$$

where $\psi^{\pm} : W^{\pm} \to \mathfrak{s}'_{\pm}$ is the restriction of ψ to $W^{\pm}$.

Proposition 4.1. *Assume the stable range condition $2n < p, q$.*

(1) *The null cone $\mathfrak{N} = \mathfrak{N}^+ \times \mathfrak{N}^-$ is an irreducible normal variety, which is of complete intersection.*

(2) *There are finitely many $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $\mathfrak{N}$, which are completely classified by the ranks of each component in $\mathfrak{N}^{\pm}$. Among them, there is an open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\mathcal{O}_{n,n}$, which is a single $K_{\mathbb{C}}$-orbit. Moreover, the singular locus coincides with $\mathfrak{N} \backslash \mathcal{O}_{n,n}$, which is of codimension ≥ 2.*

(3) *The regular function ring* $\mathbb{C}[\mathfrak{N}]$ *is isomorphic to the harmonics* $\mathcal{H}(K_{\mathbb{C}})$ *as a* $K_{\mathbb{C}} \times K'_{\mathbb{C}}$*-module. It is explicitly given as*

$$\mathbb{C}[\mathfrak{N}^+] \simeq \mathcal{H}^+(K_{\mathbb{C}}^+) \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)*} \otimes \tau_\lambda^{(n)*},$$
$$\mathbb{C}[\mathfrak{N}^-] \simeq \mathcal{H}^-(K_{\mathbb{C}}^-) \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(q)} \otimes \tau_\lambda^{(n)},$$
$$\mathbb{C}[\mathfrak{N}] \simeq \mathcal{H}(K_{\mathbb{C}}) \simeq \mathcal{H}^+(K_{\mathbb{C}}^+) \otimes \mathcal{H}^-(K_{\mathbb{C}}^-),$$

where $\mathcal{P}_n$ *denotes the set of all the partitions of length* $\leq n$*, and* $\tau_\lambda^{(n)}$ *(resp.* $\sigma_\lambda^{(p)}$*) is an irreducible finite dimensional representation of* GL_n *(resp.* $O(p, \mathbb{C})$*) of highest weight* λ*.*

Remark 4.2. (1) In [12], the orbit decomposition of the null cone is carried out for $K_{\mathbb{C}} \times L'_{\mathbb{C}}$, though it is more natural to consider $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits. In fact, their orbits are the same (under the assumption of stable range). Also, in [12], the normality is proved when $2n + 2 \leq p, q$. Here we improve it, and there is no more restriction other than the stable range condition.
(2) It is rather subtle to specify the representation $\sigma_\lambda^{(p)}$ because $O(p, \mathbb{C})$ is not connected. For this, see [8, §3.6.2].

Proof. It is straightforward to check the claim for the $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit decomposition in (2) directly by calculation. Let us denote

$$\mathcal{O}_r^+ = \{A \in M_{p,n} \mid {}^tAA = 0, \operatorname{rank} A = r\} \subset \mathfrak{N}^+,$$
$$\mathcal{O}_s^- = \{B \in M_{q,n} \mid {}^tBB = 0, \operatorname{rank} B = s\} \subset \mathfrak{N}^-,$$
$$\mathcal{O}_{r,s} = \mathcal{O}_r^+ \times \mathcal{O}_s^- \in \mathfrak{N}.$$

Then $\mathcal{O}_r^+$ is a $K_{\mathbb{C}}^+ \times K'_{\mathbb{C}}$-orbit, and $\mathcal{O}_{r,s}$ is a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit. Since they are classified by rank, the orbit of the largest possible rank $\mathcal{O}_{n,n}$ is open dense in $\mathfrak{N}$. This implies that $\mathfrak{N} = \overline{\mathcal{O}_{n,n}}$ is irreducible. In fact, $\mathcal{O}_{n,n}$ is a single $K_{\mathbb{C}}$-orbit. This follows from the Witt theorem because the orbit is of full rank.

Explicit calculation of the rank of differentials of the defining equations of $\psi^*(x_{ij})$'s and $\psi^*(y_{ij})$'s tells us that a point from $\mathfrak{N} \backslash \mathcal{O}_{n,n}$ is singular. Since

$$\dim \mathfrak{N}^+ = np - \frac{n(n+1)}{2}, \qquad \dim \mathfrak{N}^- = nq - \frac{n(n+1)}{2},$$

we have $\operatorname{codim} \mathfrak{N} = n(n+1)$, which is equal to the number of the defining equations. This proves that $\mathfrak{N}$ is of complete intersection.

The dimension of the $K_{\mathbb{C}}^+ \times K'_{\mathbb{C}}$-orbit in $\mathfrak{N}^+$ of rank r is given by

$$\dim \mathcal{O}_r^+ = r(p+n) - r^2 - \frac{r(r+1)}{2}.$$

From this formula, we can easily show that $\operatorname{codim}(\mathfrak{N} \backslash \mathcal{O}_{n,n}) \geq 2$.

By Kostant theorem (see [2, Th. 2.2.11], for example), we know that the defining radical ideal of $\mathfrak{N}$ is generated by $\psi^*(x_{ij})$'s and $\psi^*(y_{ij})$'s, which are basic invariants of $O(p,\mathbb{C}) \times O(q,\mathbb{C})$. This means that

$$\begin{aligned}\mathbb{C}[\mathfrak{N}] &= \mathbb{C}[W]/(\psi^*(x_{ij}),\psi^*(y_{ij})) \\ &= \mathbb{C}[W]/\mathbb{C}[W]\cdot\mathbb{C}[W]_+^{K_{\mathbb{C}}} \simeq \mathcal{H}(K_{\mathbb{C}}).\end{aligned}$$

The decomposition of $\mathcal{H}(K_{\mathbb{C}})$ is given in [8, Th. 3.7.3.1 & Cor. 3.7.3.6]. See also [3] and [10]. By the same theorem of Kostant, the null cone $\mathfrak{N}$ is normal. Also, $\mathfrak{N}^{\pm}$ are normal varieties. □

4.2. $U(p,q)\times U(m,n)$ $(m+n \le \min(p,q))$**.** Put $G = U(p,q)$ and $G' = U(m,n)$, and assume the stable range condition $m+n \le p,q$. We denote:

$$\begin{array}{ccc} M = U(p,q)^2 & \overset{\Delta}{\supset} & G = U(p,q) \\ \cup & & \cup \\ L = L^+ \times L^- = U(p)^2 \times U(q)^2 & \overset{\Delta}{\supset} & K = K^+ \times K^- = U(p)\times U(q) \end{array}$$

Here Δ denotes the diagonal embedding. Similarly, we denote M', G', L', K' replacing p and q by m and n respectively.

Put

$$\begin{aligned} W = M_{p+q,m+n}(\mathbb{C}) &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A \in M_{p,m}, B \in M_{p,n}, C \in M_{q,m}, D \in M_{q,n} \right\} \\ &= W^+ \oplus W^- = M_{p,m+n}(\mathbb{C}) \oplus M_{q,m+n}(\mathbb{C}). \end{aligned}$$

Then $K_{\mathbb{C}} = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ and $K'_{\mathbb{C}} = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ act on W as

$$((g_1,g_2),(h_1,h_2)) \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} g_1 A\, {}^t h_1 & {}^t g_1{}^{-1} B h_2^{-1} \\ {}^t g_2{}^{-1} C h_1^{-1} & g_2 D\, {}^t h_2 \end{pmatrix},$$

$$(g_1,g_2) \in GL_p(\mathbb{C}) \times GL_q(\mathbb{C}), \quad (h_1,h_2) \in GL_m(\mathbb{C}) \times GL_n(\mathbb{C}).$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ as

$$\mathfrak{g} = \mathfrak{gl}_{p+q}(\mathbb{C}) = \left(\begin{array}{c|c} M_p(\mathbb{C}) & 0 \\ \hline 0 & M_q(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & M_{p,q}(\mathbb{C}) \\ \hline M_{q,p}(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k} \oplus \mathfrak{s}.$$

The Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ is chosen similarly. We identify

$$\mathfrak{s} = M_{p,q}(\mathbb{C}) \oplus M_{q,p}(\mathbb{C}), \qquad \mathfrak{s}' = M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C}).$$

The moment maps φ and ψ are explicitly given by

$$\varphi : W \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto (A\,{}^tC, D\,{}^tB) \in \mathfrak{s}, \quad \psi : W \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto ({}^tAB, {}^tDC) \in \mathfrak{s}'.$$

These maps are $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant with the trivial $K_{\mathbb{C}}$-action on $\mathfrak{s}'$, and the trivial $K'_{\mathbb{C}}$-action on $\mathfrak{s}$ respectively. The maps φ and ψ are almost the same. Therefore we will treat only the map ψ in the following.

The map ψ induces an algebra homomorphism $\psi : \mathbb{C}[\mathfrak{s}'] \to \mathbb{C}[W]$ by

$$\psi^*(x_{i,j}) = \sum_{k=1}^{p} a_{k,i} b_{k,j}, \quad X = (x_{i,j}) \in M_{m,n}(\mathbb{C}),$$
$$\psi^*(y_{i,j}) = \sum_{l=1}^{q} c_{l,j} d_{l,i}, \quad Y = (y_{i,j}) \in M_{n,m}(\mathbb{C}).$$

GL_p-invariants (resp. GL_q-invariants) on $\mathbb{C}[W^+]$ (resp. $\mathbb{C}[W^-]$) are generated by $\psi(x_{i,j})^*$'s (resp. $\psi(y_{i,j})^*$'s). Hence $\psi : W \to \mathfrak{s}'$ is an affine quotient map by $K_{\mathbb{C}}$, which is surjective under the condition of the stable range. Note that φ is *not* surjective in general.

Let us consider the null cones

$$\mathfrak{N} = \psi^{-1}(0) \subset W, \qquad \mathfrak{N}^{\pm} = (\psi^{\pm})^{-1}(0) \subset W^{\pm},$$

where $\psi^{\pm} : W^{\pm} \to \mathfrak{s}'_{\pm}$ is the restriction of ψ to $W^{\pm}$.

Proposition 4.3. *Assume the stable range condition $m + n \leq \min(p, q)$.*

(1) *The null cone $\mathfrak{N} = \mathfrak{N}^+ \times \mathfrak{N}^-$ is an irreducible normal variety, which is of complete intersection.*

(2) *There are finitely many $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $\mathfrak{N}$, which are completely classified by the ranks of each component in $\mathfrak{N}^{\pm}$. Among them, there is an open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\mathcal{O}_{m,n}$, which is a single $K_{\mathbb{C}}$-orbit. Moreover, the singular locus is of codimension ≥ 2.*

(3) *The regular function ring $\mathbb{C}[\mathfrak{N}]$ is isomorphic to the harmonics $\mathcal{H}(K_{\mathbb{C}})$ as a $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-module. It is explicitly given as*

$$\mathbb{C}[\mathfrak{N}^+] \simeq \mathcal{H}^+(K_{\mathbb{C}}^+) \simeq \sum_{\alpha \in \mathcal{P}_m, \beta \in \mathcal{P}_n}^{\oplus} \tau_{\alpha \odot \beta}^{(p)\,*} \otimes \left(\tau_{\alpha}^{(m)*} \otimes \tau_{\beta}^{(n)} \right),$$
$$\mathbb{C}[\mathfrak{N}^-] \simeq \mathcal{H}^-(K_{\mathbb{C}}^-) \simeq \sum_{\gamma \in \mathcal{P}_m, \delta \in \mathcal{P}_n}^{\oplus} \tau_{\gamma \odot \delta}^{(q)} \otimes \left(\tau_{\gamma}^{(m)} \otimes \tau_{\delta}^{(n)*} \right),$$
$$\mathbb{C}[\mathfrak{N}] \simeq \mathcal{H}(K_{\mathbb{C}}) \simeq \mathcal{H}^+(K_{\mathbb{C}}^+) \otimes \mathcal{H}^-(K_{\mathbb{C}}^-),$$

where, for partitions $\alpha \in \mathcal{P}_m$ and $\beta \in \mathcal{P}_n$, we denote

$$\alpha \odot \beta = \alpha \odot_p \beta = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0, -\beta_n, \ldots, -\beta_1) \in \mathbb{Z}^p$$

and $\tau_{\lambda}^{(p)}$ is an irreducible finite dimensional representation of GL_p of highest weight λ.

Proof. Put

$$\mathcal{O}_{r,s}^+ = \{(A, B) \in M_{p,m+n} \mid {}^t AB = 0, \operatorname{rank} A = r, \operatorname{rank} B = s\} \subset \mathfrak{N}^+,$$

Then $\mathcal{O}^+_{r,s}$ is a $K^+_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit, and

$$\dim \mathcal{O}^+_{r,s} = r(m+p) + s(n+p) + rs - (r+s)^2.$$

It is easy to see that $\mathcal{O}^+_{m,n} \subset \mathfrak{N}^+$ is an open dense orbit and

$$\dim \mathfrak{N}^+ = \dim \mathcal{O}^+_{m,n} = p(m+n) - mn,$$

which means $\mathfrak{N}^+$ is of complete intersection. The singular locus is given by

$$\overline{\mathcal{O}^+_{m-1,n-1}} = \coprod_{r \le m-1, s \le n-1} \mathcal{O}^+_{r,s} \subset \mathfrak{N}^+,$$

which is of codimension ≥ 2. The rest of the proof is similar to that of Proposition 4.1.

The decomposition of harmonics is given in [8, Th. 2.5.4]. $\square$

4.3. $Sp(p,q) \times O^*(2n)$ $(n \le \min(p,q))$. Put $G = Sp(p,q)$ and $G' = O^*(2n)$, and assume the stable range condition $n \le p, q$. We denote:

$$\begin{array}{ccc} M = U(2p,2q) & \supset & G = Sp(p,q) \\ \cup & & \cup \\ L = L^+ \times L^- = U(2p) \times U(2q) & \supset & K = K^+ \times K^- = Sp(p) \times Sp(q) \end{array}$$

$$\begin{array}{ccc} M' = O^*(2n)^2 & \overset{\Delta}{\supset} & G' = O^*(2n) \\ \cup & & \cup \\ L' = K' \times K' = U(n)^2 & \overset{\Delta}{\supset} & K' = U(n) \end{array}$$

Put

$$\begin{aligned} W = M_{2p+2q,n}(\mathbb{C}) &= \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \mid A \in M_{2p,n}(\mathbb{C}), B \in M_{2q,n}(\mathbb{C}) \right\} \\ &= W^+ \oplus W^- = M_{2p,n}(\mathbb{C}) \oplus M_{2q,n}(\mathbb{C}). \end{aligned}$$

The complexification $K_{\mathbb{C}} = Sp(2p,\mathbb{C}) \times Sp(2q,\mathbb{C})$ acts on W as

$$(k_1,k_2) \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} k_1 A \\ {}^t k_2^{-1} B \end{pmatrix}, \qquad (k_1,k_2) \in Sp(2p,\mathbb{C}) \times Sp(2q,\mathbb{C}),$$

and $K'_{\mathbb{C}} = GL_n(\mathbb{C})$ acts on W as

$$g \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A\,{}^t g \\ B g^{-1} \end{pmatrix}, \qquad g \in GL_n(\mathbb{C}).$$

We fix Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ as

$$\begin{aligned} \mathfrak{g} = \mathfrak{sp}(2p+2q,\mathbb{C}) &= \left(\begin{array}{c|c} \mathfrak{sp}(2p,\mathbb{C}) & 0 \\ \hline 0 & \mathfrak{sp}(2q,\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & M_{2p,2q} \\ \hline J_q\,{}^tM_{2p,2q}J_p & 0 \end{array}\right), \\ &= \mathfrak{k} \oplus \mathfrak{s}, \quad J_p = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix} \end{aligned}$$

$$\mathfrak{g}' = \mathfrak{o}(2n,\mathbb{C}) = \left(\begin{array}{c|c} M_n(\mathbb{C}) & 0 \\ \hline 0 & -\,{}^tM_n(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & \mathrm{Alt}_n(\mathbb{C}) \\ \hline \mathrm{Alt}_n(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k}' \oplus \mathfrak{s}'.$$

Thus, we can identify $\mathfrak{s} = M_{2p,2q}(\mathbb{C})$ and $\mathfrak{s}' = \mathfrak{s}'_+ \oplus \mathfrak{s}'_- = \mathrm{Alt}_n(\mathbb{C}) \oplus \mathrm{Alt}_n(\mathbb{C})$. The moment maps are explicitly given by

$$\varphi : W \ni \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto A\,{}^tB \in \mathfrak{s}, \qquad \psi : W \ni \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto ({}^tAJ_pA,\ {}^tBJ_qB) \in \mathfrak{s}'.$$

These maps are clearly $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant with the trivial $K_{\mathbb{C}}$-action on $\mathfrak{s}'$, and the trivial $K'_{\mathbb{C}}$-action on $\mathfrak{s}$ respectively. The moment maps induce algebra morphisms φ^* and ψ^*:

$$\begin{aligned} \varphi^*(z_{i,j}) &= \sum_{k=1}^{n} a_{i,k}b_{j,k}, \quad Z = (z_{i,j}) \in M_{2p,2q}, \quad A = (a_{i,j}) \in M_{2p,n}, B = (b_{i,j}) \in M_{2q,n}, \\ \psi^*(x_{i,j}) &= -\sum_{k=1}^{p} a_{k+p,i}a_{k,j} + \sum_{k=1}^{p} a_{k,i}a_{k+p,j}, \quad X = (x_{i,j}) \in \mathrm{Alt}_n, \\ \psi^*(y_{i,j}) &= -\sum_{l=1}^{q} b_{l+q,i}b_{l,j} + \sum_{l=1}^{q} b_{l,i}b_{l+q,j}, \quad Y = (y_{i,j}) \in \mathrm{Alt}_n. \end{aligned}$$

Classical invariant theory tells us that $\varphi^*(z_{i,j})$'s generate the GL_n-invariants $\mathbb{C}[W]^{K'_{\mathbb{C}}}$, and that $\psi(x_{i,j})^*$'s (resp. $\psi(y_{i,j})^*$'s) generate the $K^+_{\mathbb{C}} = Sp(2p,\mathbb{C})$-invariants $\mathbb{C}[W^+]^{K^+_{\mathbb{C}}}$ (resp. $K^-_{\mathbb{C}} = Sp(2q,\mathbb{C})$-invariants $\mathbb{C}[W^-]^{K^-_{\mathbb{C}}}$). This shows that φ and ψ are affine quotient maps. By the assumption of the stable range, $n \leq p,q$, the map $\psi : W \to \mathfrak{s}'$ is a surjection, and we have

$$\mathbb{C}[\mathfrak{s}'] \simeq \mathbb{C}[W]^{K_{\mathbb{C}}}, \qquad \mathbb{C}[\mathfrak{s}'_\pm] \simeq \mathbb{C}[W^\pm]^{K^\pm_{\mathbb{C}}}.$$

Let us define the null cones by

$$\mathfrak{N} = \psi^{-1}(0) \subset W, \qquad \mathfrak{N}^\pm = (\psi^\pm)^{-1}(0) \subset W^\pm,$$

where $\psi^\pm : W^\pm \to \mathfrak{s}'_\pm$ is the restriction of ψ to $W^\pm$.

Proposition 4.4. *Assume the stable range condition $n \leq p,q$.*

(1) *The null cone* $\mathfrak{N} = \mathfrak{N}^+ \times \mathfrak{N}^-$ *is an irreducible normal variety, which is of complete intersection.*
(2) *There are finitely many* $K_{\mathbb{C}} \times K'_{\mathbb{C}}$*-orbits in* $\mathfrak{N}$*, which are completely classified by the ranks of each component in* $\mathfrak{N}^{\pm}$*. The full rank orbit* $\mathcal{O}_{n,n}$ *is an open dense* $K_{\mathbb{C}} \times K'_{\mathbb{C}}$*-orbit in* $\mathfrak{N}$*, which is also a single* $K_{\mathbb{C}}$*-orbit. The singular locus of* $\mathfrak{N}$ *is of codimension* ≥ 2*.*
(3) *The regular function ring* $\mathbb{C}[\mathfrak{N}]$ *is isomorphic to the harmonics* $\mathcal{H}(K_{\mathbb{C}})$ *as a* $K_{\mathbb{C}} \times K'_{\mathbb{C}}$*-module, and we have*

$$\begin{aligned}
\mathbb{C}[\mathfrak{N}^+] &\simeq \mathcal{H}^+(K_{\mathbb{C}}^+) \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)*} \otimes \tau_\lambda^{(n)*}, \\
\mathbb{C}[\mathfrak{N}^-] &\simeq \mathcal{H}^-(K_{\mathbb{C}}^-) \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(q)} \otimes \tau_\lambda^{(n)}, \\
\mathbb{C}[\mathfrak{N}] &\simeq \mathcal{H}(K_{\mathbb{C}}) \simeq \mathcal{H}^+(K_{\mathbb{C}}^+) \otimes \mathcal{H}^-(K_{\mathbb{C}}^-),
\end{aligned}$$

where $\sigma_\lambda^{(p)}$ *denotes an irreducible finite dimensional representation of* $Sp(2p, \mathbb{C})$ *with highest weight* $\lambda \in \mathcal{P}_n$*.*

Proof. Put

$$\begin{aligned}
\mathcal{O}_r^+ &= \{A \in M_{2p,n} \mid {}^tAJ_pA = 0, \operatorname{rank} A = r\} \subset \mathfrak{N}^+, \\
\mathcal{O}_s^- &= \{B \in M_{2q,n} \mid {}^tBJ_qB = 0, \operatorname{rank} B = s\} \subset \mathfrak{N}^-.
\end{aligned}$$

Then $\mathcal{O}_{r,s} = \mathcal{O}_r^+ \times \mathcal{O}_s^-$ is a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit and

$$\dim \mathcal{O}_r^+ = r(2p+n) - r^2 - \frac{r(r-1)}{2}, \qquad \dim \mathcal{O}_s^+ = s(2q+n) - s^2 - \frac{s(s-1)}{2}.$$

It is easy to show that $\mathcal{O}_n^{\pm} \subset \mathfrak{N}^{\pm}$ is an open dense orbit and

$$\dim \mathfrak{N}^+ = \dim \mathcal{O}_n^+ = 2pn - \frac{n(n-1)}{2}, \qquad \dim \mathfrak{N}^- = \dim \mathcal{O}_n^- = 2qn - \frac{n(n-1)}{2},$$

which tells us that $\mathfrak{N}$ (resp. $\mathfrak{N}^{\pm}$) is of complete intersection. The singular locus is given by

$$\coprod_{r \leq n-2} \mathcal{O}_r^{\pm} \subset \mathfrak{N}^{\pm},$$

which is of codimension ≥ 2. The rest of the proof is similar to that of Proposition 4.1.

The structure of harmonics is given in [8, Th. 3.8.6.2]. $\square$

References

[1] Walter Borho and Hanspeter Kraft, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen. Comment. Math. Helv. 54 (1979), no. 1, 61–104.

[2] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Boston, MA, 1997 ISBN: 0-8176-3792-3.

[3] Stephen S. Gelbart, A theory of Stiefel harmonics. Trans. Amer. Math. Soc. 192 (1974), 29–50.

[4] Robin Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[5] R. Howe, θ-series and invariant theory, Proc. Symp. Pure Math. **33**, Ameri. Math. Soc., Providence, 1979.

[6] R. Howe, Remarks on classical invariant theory. Trans. Amer. Math. Soc. 313 (1989), no. 2, 539–570. Erratum to: "Remarks on classical invariant theory". Trans. Amer. Math. Soc. 318 (1990), no. 2, 823.

[7] R. Howe, Transcending classical invariant theory. J. Amer. Math. Soc. 2 (1989), no. 3, 535–552.

[8] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. in *The Schur lectures (1992) (Tel Aviv)*, 1–182, Bar-Ilan Univ., Ramat Gan, 1995.

[9] J.-S. Huang and J.-S. Li, Unipotent representations attached to spherical nilpotent orbits, Amer. Jour. Math. **121** (1999), no. 3, 497–517.

[10] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math. **44** (1978), no. 1, 1–47.

[11] J.-S. Li, Singular unitary representations of classical groups, Invent. Math., **97** (1989), 237 – 255.

[12] Kyo Nishiyama, Multiplicity-free actions and the geometry of nilpotent orbits. Math. Ann., **318** (2000), 777 – 793.

[13] Kyo Nishiyama, Theta lifting of two-step nilpotent orbits for the pair $O(p,q) \times Sp(2n,\mathbb{R})$. In *"Infinite Dimensional Harmonic Analysis"*, Transaction of a Japanese-German Symposium (Kyoto, September 1999), Kyoto 1999, pp. 278 – 289.

[14] Kyo Nishiyama, Hiroyuki Ochiai, and Kenji Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules—Hermitian symmetric case. Astérisque No. 273, (2001), 13–80.

[15] Kyo Nishiyama and C.-B. Zhu, Theta lifting of holomorphic discrete series, (The case of $U(p,q) \times U(n,n)$). Trans. AMS. **353** (2001), 3327-3345.

[16] Takuya Ohta, Nilpotent orbits of Z_4-graded Lie algebra and geometry of the moment maps associated to the dual pair $(U(p,q), U(r,s))$, preprint.

[17] Takuya Ohta, Nilpotent orbits of Z_4-graded Lie algebra and geometry of the moment maps associated to the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$ and $(O^*(2n), Sp(p,q))$, in preparation.

[18] V. L. Popov and E. B. Vinberg, Invariant theory, in "Algebraic geometry. IV.", Encyclopaedia of Mathematical Sciences, 55. Springer-Verlag, Berlin, 1994, pp. 123–284.

[19] T. Przebinda, Characters, dual pairs, and unitary representations, Duke Math. J. **69** (1993), no. 3, 547-594.

[20] E. B. Vinberg and V. L. Popov, A certain class of quasihomogeneous affine varieties. Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972), 749–764.

[21] H. Weyl, The classical groups, Princeton University Press, 1946.

Faculty of IHS, Kyoto University, Sakyo, Kyoto 606-8501, Japan
E-mail address: kyo@math.h.kyoto-u.ac.jp

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260
E-mail address: matzhucb@nus.edu.sg