Associated Varieties and Gelfand-Kirillov Dimensions for the Discrete Series of a Semisimple Lie Group

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Introduction.

Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding complexified Lie algebras are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. We assume Harish-Chandra's rank condition $\operatorname{rank} G = \operatorname{rank} K$, which is necessary and sufficient for G to have a non-empty set of discrete series, consisting of square-integrable irreducible unitary representations of G ([6]).

Concrete geometric realizations of discrete series representations have been obtained in several ways (see e.g., the survey article [5]). Among others, Hotta and Parthasarathy [9] realize such representations on the kernel spaces of certain G-invariant differential operators $\mathcal{D}_\lambda$ of gradient-type, defined on vector bundles over the symmetric space G/K, by using some elementary differential calculas on G/K (see §5). Here λ denotes the lowest highest weight of corresponding discrete series. As we have shown in [20], the operators $\mathcal{D}_\lambda$ allow us to determine the embeddings of discrete series into various important induced G-modules.

In this paper, we describe the associated varieties of Harish-Chandra $(\mathfrak{g}, K)$-modules of discrete series, by quite an elementary method based on the above work of Hotta-Parthasarathy. Our description is as in

THEOREM. (Theorem 3.1) *If H_Λ is the $(\mathfrak{g}, K)$-module of discrete series with Harish-Chanda parameter $\Lambda = \lambda + \rho_c - \rho_n$ (see §2), then its associated variety $\mathcal{V}(H_\Lambda) \subset \mathfrak{g}$ (see §1 for the definition) coincides with the nilpotent cone $K_C\mathfrak{p}_-$. Here K_C is the analytic subgroup of adjoint group $G_C := \operatorname{Int}(\mathfrak{g})$ of $\mathfrak{g}$, with Lie algebra $\mathfrak{k}$, and $\mathfrak{p}_-$ denotes the sum of root subspaces of $\mathfrak{g}$ corresponding to the non-compact roots which are negative with respect to Λ.*

This theorem enables us to deduce that the variety $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$ associated to primitive ideal $I_\Lambda := \operatorname{Ann}_{U(\mathfrak{g})}(H_\Lambda)$ in the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is just the closure of the cone $G_C\mathfrak{p}_-$ (Theorem 3.2). We can further derive explicit (recursion) formulae for the Gelfand-Kirillov dimensions $d(H_\Lambda) := \dim \mathcal{V}(H_\Lambda)$ of discrete series in the case of unitary groups $G = SU(p, q)$ (Theorems 8.2 and 8.3), by specifying the unique nilpotent G_C-orbit in $\mathfrak{g}$ which intersects $\mathfrak{p}_-$ densely.

To prove the above Theorem 3.1 we first assume that the parameter λ is sufficiently regular, and we pass to a graded space of coefficients of Taylor expansions of analytic sections in $\mathrm{Ker}\,\mathcal{D}_\lambda$. This space of coefficients, say $\mathrm{Gr}(\mathrm{Ker}\,\mathcal{D}_\lambda)$, admits a natural $S(\mathfrak{g})$-module structure, where $S(\mathfrak{g}) \simeq \mathrm{gr}\,U(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$. Then it can be shown by using Theorem 1 of [9] that the corresponding annihilator ideal in $S(\mathfrak{g})$ defines the associated variety of dual discrete series module H_Λ^* as the set of common zero points. Another point in proving our theorem is that the $S(\mathfrak{g})$-module $\mathrm{Gr}(\mathrm{Ker}\,\mathcal{D}_\lambda)$ is characterized as the kernel of a differential operator $\mathrm{Gr}[\mathcal{D}_\lambda]$ defined on a space of (vector-valued) polynomial functions on $\mathfrak{p}$, which naturally arises from $\mathcal{D}_\lambda$ by passing to the gradation ([9], see Theorem 5.1).

These two results allow us to establish Theorem 3.1 for sufficiently regular λ, by examining in §6 the annihilator ideal of $S(\mathfrak{g})$-module $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ more closely. We further find that for such a λ the ideal $\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ coincides with its radical in $S(\mathfrak{g})$ (Corollary 6.1). Finally, our theorem for arbitrary λ follows from the result for the above sufficiently regular case, with in mind Zuckerman's translation principle (see §7).

One may say that the above description of $\mathcal{V}(H_\Lambda)$ is known among the experts in D-module theory. This is because: (a) the associated variety of a Harish-Chandra module is gained, through the moment map, as the image of characteristic variety of corresponding D-module over the complexified flag variety X of G, and (b) the characteristic variety of a discrete series D-module can be specified as the conormal bundle of a closed K_C-orbit on X. (See [3, III] and [13].)

However these (a) and (b) rely on several deep results about the classification of irreducible G-representations through D-modules, K_C-orbit structure of the variety X, etc., although the associated variety is a very simple object defined for each finitely generated $U(\mathfrak{g})$-module in a purely algebraic context (see §1). Moreover, no self-contained and concise references for Theorem 3.1 are easily available up to the present.

From these reasons, we make here a short-cut and describe directly our variety $\mathcal{V}(H_\Lambda)$ only by using some basic facts on realization of discrete series. Here are placed our motivation and emphasis of this presentation.

The organization of this paper is as follows. We begin with introducing in §§1-2 three principal objects of our concern: the associated variety, Gelfand-Kirillov dimension for $U(\mathfrak{g})$-module; and the discrete series for G. In §3, our main theorem for the variety $\mathcal{V}(H_\Lambda)$ is given as Theorem 3.1, and then we deduce from it two important consequences (Theorem 3.2 and Proposition 3.2). The succeeding four sections, §§4-7, are devoted to proving Theorem 3.1, where we are based on the excellent work [9]. The last section, §8, specifies the varieties $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$, and gives explicit formulae for the Gelfand-Kirillov dimensions $d(H_\Lambda)$. We concentrate on the groups $G = SU(p,q)$, where p and q range

over non-negative integers such that $(p,q) \neq (0,0)$. Our formula obtained in Theorem 8.3 is recursive with respect to the parameter $n = p + q$.

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1. Associated varieties for $U(\mathfrak{g})$-modules.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We begin with introducing two important invariants: the associated variety and Gelfand-Kirillov dimension, for finitely generated $U(\mathfrak{g})$-modules.

Denote by $(U_k(\mathfrak{g}))_{k=0,1,..}$ the natural increasing filtration of $U(\mathfrak{g})$, where $U_k(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \cdots X_m$ $(m \leq k)$ with $X_j \in \mathfrak{g}$ $(1 \leq j \leq m)$. By the Poincaré-Birkhoff-Witt theorem, we can and do identify the associated graded ring

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{k \geq 0} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g}) \quad (U_{-1}(\mathfrak{g}) := (0))$$

with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k \geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. Here $S^k(\mathfrak{g})$ denotes the homogeneous component of $S(\mathfrak{g})$ of degree k.

Let H be a finitely generated $U(\mathfrak{g})$-module. Take a finite-dimensional subspace H_0 of H such that $H = U(\mathfrak{g})H_0$. Setting $H_k = U_k(\mathfrak{g})H_0$ $(k = 1, 2, ...)$, one gets an increasing filtration $(H_k)_k$ of H and correspondingly a finitely generated, graded $S(\mathfrak{g})$-module

$$M = \operatorname{gr}(H; H_0) := \bigoplus_{k \geq 0} M_k \quad \text{with } M_k = H_k/H_{k-1}. \tag{1.1}$$

The annihilator $\operatorname{Ann}_{S(\mathfrak{g})} M := \{D \in S(\mathfrak{g}) |\ Dv = 0\ (\forall v \in M)\}$ of M is a graded ideal of $S(\mathfrak{g})$, and it defines an algebraic cone in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$:

$$\mathcal{V}(M) := \{\lambda \in \mathfrak{g}^* |\ f(\lambda) = 0\ (\forall f \in \operatorname{Ann}_{S(\mathfrak{g})} M)\},$$

as the set of common zeros of elements of $\operatorname{Ann}_{S(\mathfrak{g})} M$. Here $S(\mathfrak{g})$ is looked upon as the polynomial ring over $\mathfrak{g}^*$ in the canonical way. It is then easily seen that the variety $\mathcal{V}(M)$ does not depend on the choice of a generating subspace H_0. So, hereafter we write $\mathcal{V}(H)$ for this invariant $\mathcal{V}(M)$ of H.

DEFINITION.(Cf. [16], [21]) For a finitely generated $U(\mathfrak{g})$-module H, the variety $\mathcal{V}(H) \subset \mathfrak{g}^*$ and its dimension $d(H) := \dim \mathcal{V}(H)$ are called respectively the *associated variety* and the *Gelfand-Kirillov dimension* of H.

It should be noticed that, by the Hilbert-Serre theorem (cf. [21, Th.1.1]), the map $k \to \dim H_k$ coincides with a polynomial in k of degree $d(H)$, for sufficiently large k.

2. Discrete series for a semisimple Lie group.

Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding Lie algebras are denoted respectively by $\mathfrak{g}_0$ and $\mathfrak{k}_0$. Then one has a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$. We always assume the rank condition $\operatorname{rank} G = \operatorname{rank} K$, which is necessary and sufficient for G to have a non-empty discrete series. In this section we collect some basic facts and fix notations on the discrete series representations of G.

Take a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$, which is a compact Cartan subalgebra of $\mathfrak{g}_0$ by the above assumption on G. Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$, and we write $\mathfrak{h} \subset \mathfrak{g}$ for the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by dropping the subscript '0'. If $\alpha \in \mathfrak{t}^*$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{t}$, the corresponding root subspace

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} |\ [Z, X] = \alpha(Z)X \quad (\forall Z \in \mathfrak{t})\}$$

is contained either in $\mathfrak{k}$ or in $\mathfrak{p}$. A root α is said to be compact or non-compact according as $\mathfrak{g}_\alpha \subset \mathfrak{k}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}$. We denote the totality of roots (resp. compact roots, non-compact roots) by Δ (resp. Δ_c, Δ_n).

Now fix a positive system Δ_c^+ of Δ_c. Let Ξ be the set of linear forms Λ on $\mathfrak{t}$ satisfying the following three conditions:

(2.1) $(\Lambda, \alpha) \geq 0$ for any $\alpha \in \Delta_c^+$, i.e., Λ is Δ_c^+-dominant,

(2.2) $(\Lambda, \alpha) \neq 0$ for any $\alpha \in \Delta$, i.e., Λ is Δ-regular,

(2.3) the map $Z \to \exp < \Lambda + \rho, Z >$ $(Z \in \mathfrak{t}_0)$ defines a well-defined unitary character of the Cartan subgroup $T := \exp \mathfrak{t}_0$, i.e., $\Lambda + \rho$ is T-integral.

Here $(\cdot, \cdot)$ denotes the bilinear form on $\mathfrak{t}^*$ induced canonically from the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$, and ρ is half the sum of positive roots in Δ with respect to any fixed positive system of Δ. Notice that the condition (2.3) does not depend on the choice of a positive system which defines ρ.

By Harish-Chandra, there exists a bijective correspondance, say $\Lambda \to \pi_\Lambda$, from Ξ onto the set of (equivalence classes) of discrete series representations of G (see e.g., [20, I, Prop.1.1]). We say that the discrete series representation π_Λ has *Harish-Chandra parameter* Λ.

What is more important in this article is however the lowest K-type property which characterizes the discrete series π_Λ. To be precise, for a Δ_c^+-dominant, T-integral linear form $\mu \in \mathfrak{t}^*$, let (τ_μ, V_μ) denote the finite-dimensional irreducible K-module with highest weight μ. We set for a $\Lambda \in \Xi$,

$$\lambda := \Lambda - \rho_c + \rho_n = (\Lambda - 2\rho_c) + \rho = (\Lambda + 2\rho_n) - \rho, \tag{2.4}$$

where half the sum ρ of positive roots is defined by the positive system $\Delta^+ := \{\alpha \in \Delta |\ (\Lambda, \alpha) > 0\}$, and $\rho_c := (1/2) \cdot \sum_{\alpha \in \Delta_c^+} \alpha, \quad \rho_n := \rho - \rho_c$.

PROPOSITION 2.1. (See e.g., [5]) (i) *The discrete series representation π_Λ, looked upon as a K-module, has lowest K-type τ_λ:*

(a) *π_Λ contains τ_λ with multiplicity one,*

(b) *the highest weight of any irreducible K-representation occuring in π_Λ is of the form*

$$\lambda + \sum_{\alpha \in \Delta^+} n_\alpha \alpha$$

with non-negative integers n_α.

(ii) *Conversely, if an irreducible unitary representation π of G satisfies* (a) *and* (b), *then π is unitarily equivalent to π_Λ.*

Suggested by this proposition, we call $\lambda = \Lambda - \rho_c + \rho_n$ the *lowest highest weight* (or the *Blattner parameter*) of π_Λ.

3. Description of the associated varieties for discrete series.

We now present the main result (Theorem 3.1) of this paper, and deduce from it two important consequences (Theorem 3.2 and Proposition 3.2) concerning the associated varieties and Gelfand-Kirillov dimensions for the discrete series.

3.1. Varieties $\mathcal{V}(H_\Lambda)$ and $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$.

For a $\Lambda \in \Xi$, let H_Λ be the Harish-Chandra $(\mathfrak{g}, K)$-module corresponding to π_Λ, which is gained by passing to the K-finite part of π_Λ. It follows that H_Λ is irreducible as a $U(\mathfrak{g})$-module because of the irreducibility of the corresponding G-representation π_Λ. See e.g., [19, I, 2.4] for the definition and basic facts on Harish-Chandra $(\mathfrak{g}, K)$-modules.

Now we put

$$\mathfrak{p}_\pm := \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{\pm\alpha}, \tag{3.1}$$

where $\Delta_n^+ = \{\alpha \in \Delta_n |\ \ (\Lambda, \alpha) > 0\}$ denotes the set of non-compact positive roots with respect to Λ. Notice that the subspaces $\mathfrak{p}_\pm$ depend only on the chamber in which the Harish-Chandra parameter Λ lives. Let G_C be the adjoint group of $\mathfrak{g}$, and K_C be the analytic subgroup of G_C corresponding to the Lie subalgebra $\mathfrak{k}$.

We can describe the associated variety $\mathcal{V}(H_\Lambda)$ of H_Λ by means of the subspace $\mathfrak{p}_-$, as in

THEOREM 3.1. *The associated variety $\mathcal{V}(H_\Lambda)$ of discrete series Harish-Chandra module H_Λ coincides with the nilpotent cone $K_C\mathfrak{p}_-$. Here $\mathcal{V}(H_\Lambda)$ is regarded as a variety in $\mathfrak{g}$ by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$.*

We will prove this theorem in the succeeding sections, §§4-7, by using the gradient-type differential operators $\mathcal{D}_\lambda$ on G/K whose kernels realize the discrete series representations of G (cf. [9]).

The above theorem allows us to describe also the variety $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$ associated to the primitive ideal $I_\Lambda := \mathrm{Ann}_{U(\mathfrak{g})}H_\Lambda$, as follows.

THEOREM 3.2. *One has the equality $\mathcal{V}(U(\mathfrak{g})/I_\Lambda) = \overline{G_C\mathfrak{p}_-}$, where $\overline{A}$ denotes the Zariski closure of a subset A of $\mathfrak{g}$, and $U(\mathfrak{g})$ acts on $U(\mathfrak{g})/I_\Lambda$ by left multiplication.*

This is a direct consequence of Theorem 3.1 together with the following proposition.

PROPOSITION 3.1. *Let H be an irreducible $(\mathfrak{g}, K)$-module and $I = \mathrm{Ann}_{U(\mathfrak{g})}H$ be the corresponding primitive ideal of $U(\mathfrak{g})$. Then variety $\mathcal{V}_I := \mathcal{V}(U(\mathfrak{g})/I)$ is related to the associated variety $\mathcal{V}(H)$ of H as*

$$\mathcal{V}_I = \overline{G_C\mathcal{V}(H)}. \tag{3.2}$$

3.2. The proof of Proposition 3.1 requires four fundamental facts concerning the nilpotent G_C- or K_C-orbits, associated varieties and primitive ideals, which we are going to list up.

LEMMA 3.1. (Cf. [21, Lemma 3.1]) *Let $\mathcal{N}$ be the variety of all nilpotent elements of $\mathfrak{g}$, and put $\mathcal{N}(\mathfrak{p}) := \mathcal{N} \cap \mathfrak{p}$. If H and $I = \mathrm{Ann}_{U(\mathfrak{g})}H$ are as in Proposition 3.1, the variety $\mathcal{V}_I$ (resp. $\mathcal{V}(H)$) is a G_C-stable (resp. K_C-stable) cone contained in $\mathcal{N}$ (resp. in $\mathcal{V}_I \cap \mathfrak{p} \subset \mathcal{N}(\mathfrak{p})$).*

LEMMA 3.2. (Joseph, cf. [15, Th.3.1]) *For the above H and I, one has the equality $\dim \mathcal{V}_I = 2 \dim \mathcal{V}(H)$.*

LEMMA 3.3. (See e.g., [3, III, §4]) *The variety $\mathcal{V}_I$ associated to a primitive ideal $I = \mathrm{Ann}_{U(\mathfrak{g})}H \subset U(\mathfrak{g})$ is the closure of a single nilpotent G_C-orbit $\mathcal{O}_1$ in $\mathfrak{g}$: $\mathcal{V}_I = \overline{\mathcal{O}_1}$.*

LEMMA 3.4. (Kostant-Rallis, cf. [11, Prop.5]) *If $\mathcal{O}$ is a nilpotent K_C-orbit in $\mathfrak{p}$, the dimension of G_C-orbit $\mathcal{O}_1 := G_C\mathcal{O}$ containing $\mathcal{O}$, is equal to $2 \dim \mathcal{O}$.*

PROOF OF PROPOSITION 3.1. The inclusion $\overline{G_C\mathcal{V}(H)} \subset \mathcal{V}_I$ in (3.2) is clear from Lemma 3.1. To show the converse inclusion, take a nilpotent K_C-orbit $\mathcal{O}$ in $\mathfrak{p}$ such that $\dim \mathcal{V}(H) = \dim \mathcal{O}$. Such an $\mathcal{O}$ actually exists since the number of nilpotent K_C-orbits in $\mathfrak{p}$ is finite (see [7, Chap.III, Th.4.8]). Set $\mathcal{O}_1 = G_C\mathcal{O}$ $(\subset \mathcal{V}_I)$. Then it follows from Lemmas 3.2 and 3.4 that

$$\dim \mathcal{O}_1 = 2 \dim \mathcal{O} = 2 \dim \mathcal{V}(H) = \dim \mathcal{V}_I.$$

Hence $\mathcal{O}_1$ is an open subset of $\mathcal{V}_I$. By virtue of Lemma 3.3, we conclude that $\mathcal{V}_I = \overline{\mathcal{O}_1} \subset \overline{G_C\mathcal{V}(H)}$. This completes the proof of Proposition 3.1. Q.E.D.

REMARK. The varieties $\mathcal{V}(H)$, $\mathcal{V}_I$ are closely related to the asymptotic support and wave front set of the distribution character of H ([1]; see also [18]).

3.3. Theorem 3.2, combined with Lemmas 3.2 and 3.3, gives the following proposition, which will be useful for computing explicitly the Gelfand-Kirillov dimensions of the discrete series (see §8).

PROPOSITION 3.2. *For a $\Lambda \in \Xi$, define a subspace $\mathfrak{p}_- \subset \mathcal{N}(\mathfrak{p})$ as in* (3.1).

(i) *If $\Omega_{\mathfrak{p}_-}$ denotes the set of nilpotent G_C-orbits $\mathcal{O}_1$ in $\mathfrak{g}$ such that $\mathcal{O}_1 \cap \mathfrak{p}_- \neq \emptyset$, there exists a unique orbit $\mathcal{O}_{\mathfrak{p}_-} \in \Omega_{\mathfrak{p}_-}$ for which $\overline{\mathcal{O}_{\mathfrak{p}_-}} \supset \mathcal{O}_1$ holds for any $\mathcal{O}_1 \in \Omega_{\mathfrak{p}_-}$.*

(ii) *The Gelfand-Kirillov dimension $d(H_\Lambda)$ of discrete series $U(\mathfrak{g})$-module H_Λ coincides with $(1/2)\dim \mathcal{O}_{\mathfrak{p}_-}$.*

4. Associated varieties and realization of Harish-Chandra modules on G/K.

For a finite-dimensional representation (τ, V_τ) of K, let $\mathcal{A}(\tau)$ be the space of real analytic functions $f : G \to V_\tau$ satisfying

$$f(gk) = \tau(k)^{-1} f(g) \quad (g \in G, k \in K). \tag{4.1}$$

The group G acts on $\mathcal{A}(\tau)$ by left translation, and $\mathcal{A}(\tau)$ turns to be a $U(\mathfrak{g})$-module through differentiation. We call $\mathcal{A}(\tau)$ the G- and $U(\mathfrak{g})$-module analytically induced from τ.

This section develops a general method for describing the associated variety $\mathcal{V}(H)$ of a Harish-Chandra module H in relation with a realization in $\mathcal{A}(\tau)$ of the K-finite dual module H^*. This is a preliminary step for the proof of Theorem 3.1.

4.1. $(S(\mathfrak{g}), K)$**-module** $\mathrm{Gr}\,\mathcal{A}(\tau)$.

At first, we define subspaces $\mathcal{A}_{(k)}$ $(k \in \mathbf{Z})$ of $\mathcal{A}(\tau)$ by

$$\mathcal{A}_{(k)} := \{f \in \mathcal{A}(\tau) |\ \ (X^m f)(1) = (0)\ \ (\forall X \in \mathfrak{p},\ 0 \leq \forall m \leq k)\} \tag{4.2}$$

for $k \geq 0$, and $\mathcal{A}_{(k)} := \mathcal{A}(\tau)$ for $k < 0$, where 1 denotes the identity element of G. Then $(\mathcal{A}_{(k)})_{k \in \mathbf{Z}}$ is a decreasing filtration of $\mathcal{A}(\tau)$ such that

$$\text{each } \mathcal{A}_{(k)} \text{ is a } K\text{-stable subspace of } \mathcal{A}(\tau), \tag{4.3}$$

$$\dim \mathcal{A}(\tau)/\mathcal{A}_{(k)} < \infty \ \text{ and } \ \bigcap_k \mathcal{A}_{(k)} = (0), \tag{4.4}$$

$$U_m(\mathfrak{g})\mathcal{A}_{(k)} \subset \mathcal{A}_{(k-m)} \ \text{ for all integers } k, m \geq 0. \tag{4.5}$$

Correspondingly, one obtains a graded $S(\mathfrak{g})$-module

$$\mathrm{Gr}\,\mathcal{A}(\tau) := \bigoplus_k \mathcal{A}_{(k)}/\mathcal{A}_{(k+1)}, \tag{4.6}$$

which admits by (4.3) a K-module structure, compatible with the $S(\mathfrak{g})$-action.

It is not difficult to analyze this $(S(\mathfrak{g}), K)$-module. To do this, let $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ be two bases of the vector space $\mathfrak{p}$ such that $B(X_i, X_j^*) = \delta_j^i$ (Kronecker's δ) for the Killing form B of $\mathfrak{g}$. We put

$$\iota_k(f) := \sum_{|\nu|=k+1} \frac{1}{\nu!}(X^*)^\nu \otimes (X^\nu f)(1) \in S^{k+1}(\mathfrak{p}) \otimes V_\tau \ \ (f \in \mathcal{A}_{(k)}), \tag{4.7}$$

where $X^\nu := X_1^{\nu_1} \cdots X_s^{\nu_s}$, $(X^*)^\nu := (X_1^*)^{\nu_1} \cdots (X_s^*)^{\nu_s}$ and $\nu! = \nu_1! \cdots \nu_s!$ for multi-indices $\nu = (\nu_1, \ldots, \nu_s)$ of length $|\nu| := \nu_1 + \cdots + \nu_s = k + 1$. Observe that the assignment $\mathcal{A}_{(k)} \ni f \to \iota_k(f) \in S^{k+1}(\mathfrak{p}) \otimes V_\tau$ is independent of the choice of of dual bases $(X_i)_i$ and $(X_i^*)_i$, and ι_k naturally gives rise to a K-isomorphism:

$$\tilde{\iota}_k : \mathcal{A}_{(k)}/\mathcal{A}_{(k+1)} \simeq S^{k+1}(\mathfrak{p}) \otimes V_\tau, \tag{4.8}$$

where $S^{k+1}(\mathfrak{p})$ is looked upon as a K-module by the adjoint action.

Through the Killing form B, we identify the symmetric algebra $S(\mathfrak{p}) = \bigoplus_{k \geq 0} S^k(\mathfrak{p})$ of $\mathfrak{p}$ with the ring generated by polynomial functions on $\mathfrak{g}$ which vanish identically on $\mathfrak{k}$. Let $S(\mathfrak{g})$ act on $S(\mathfrak{p})$ canonically as the ring of constant coefficient differential operators on the vector space $\mathfrak{g}$. It should be noticed that the action of a $Y \in \mathfrak{g}$ on $X^m \in S(\mathfrak{p})$ with $X \in \mathfrak{p}$, $m = 1, 2, \ldots$, is given by

$$Y \cdot X^m = mB(X, Y)X^{m-1}, \tag{4.9}$$

since one has for $Z \in \mathfrak{g}$,

$$\begin{aligned}(Y \cdot X^m)(Z) &= \frac{d}{dt} X^m(Z+tY)|_{t=0} \\ &= \frac{d}{dt} B(X, Z+tY)^m|_{t=0} \\ &= m B(X,Y) B(X,Z)^{m-1} \\ &= (m B(X,Y) \cdot X^{m-1})(Z).\end{aligned}$$

Summing up the isomorphisms $\tilde{\iota}_k$ ($k \in \mathbf{Z}$) in (4.8), one obtains the following lemma which visualizes the structure of $\mathrm{Gr}\,\mathcal{A}(\tau)$.

LEMMA 4.1. *The map $\tilde{\iota} := \oplus_k \tilde{\iota}_k$ gives a graded $(S(\mathfrak{g}), K)$-module isomorphism from $\mathrm{Gr}\,\mathcal{A}(\tau)$ onto the tensor product $S(\mathfrak{p}) \otimes V_\tau$, where $S(\mathfrak{g})$ acts on V_τ trivially.*

PROOF. Since each $\tilde{\iota}_k$ is a K-isomorphism, the sum $\tilde{\iota}$ commutes with the K-action. So we only have to prove that

$$\tilde{\iota}_{k-1}(Y \cdot (f + \mathcal{A}_{(k+1)})) = Y \cdot \tilde{\iota}_k(f + \mathcal{A}_{(k+1)}) \tag{4.10}$$

for $f \in \mathcal{A}_{(k)}$ and $Y \in \mathfrak{g}$. Writing Y by means of the basis $(X_i)_i$ of $\mathfrak{p}$ as

$$Y \equiv \sum_{i=1}^{s} B(X_i^*, Y) X_i \mod \mathfrak{k},$$

one sees that

$$Y \cdot (f + \mathcal{A}_{(k+1)}) = \sum_{i=1}^{s} B(X_i^*, Y) X_i f + \mathcal{A}_{(k)}, \tag{4.11}$$

since $\mathfrak{k} \cdot \mathcal{A}_{(k)} \subset \mathcal{A}_{(k)}$. With (4.9) in mind, the left hand side of (4.10) is calculated as

$$\begin{aligned}\tilde{\iota}_{k-1}(Y \cdot (f + \mathcal{A}_{(k+1)})) &= \sum_{|\nu|=k} \frac{1}{\nu!} (X^*)^\nu \otimes (\sum_i X^\nu B(X_i^*, Y) X_i f)(1) \\ &= \sum_{|\nu|=k} \sum_i \frac{1}{\nu!} B(X_i^*, Y) (X^*)^\nu \otimes ((X^\nu X_i) f)(1) \\ &= \sum_{|\nu|=k} \sum_i \frac{1}{(\nu+e_i)!} Y \cdot (X^*)^{\nu+e_i} \otimes (X^{\nu+e_i} f)(1) \\ &= \sum_{|\nu|=k+1} \frac{1}{\nu!} Y \cdot (X^*)^\nu \otimes (X^\nu f)(1) \\ &= Y \cdot \tilde{\iota}_k(f + \mathcal{A}_{(k+1)}),\end{aligned}$$

where $e_i := (\delta_j^i)_{1 \le j \le s}$, and $((X^\nu X_i) f)(1) = (X^{\nu+e_i} f)(1)$ holds since $f \in \mathcal{A}_{(k)}$. We thus get the lemma. Q.E.D.

4.2. Variety $\mathcal{V}(H)$ in relation with $\mathrm{Gr}_\xi(H^*)$.

Now let H be an irreducible $(\mathfrak{g}, K)$-module. Then the full dual space H' of H, consisting of all linear forms on H, has a $(\mathfrak{g}, K)$-module structure contragredient to H. The K-finite part of H', say H^*, is an irreducible $(\mathfrak{g}, K)$-submodule of H'.

If (τ, V_τ) is a finite-dimensional K-module occuring in H^*, there exists, by a reciprocity theorem of Frobenius type, a $(\mathfrak{g}, K)$-module embedding ξ from H^* into the analytically induced module $\mathcal{A}(\tau)$. Setting

$$H^*_{(k),\,\xi} := \xi(H^*) \cap \mathcal{A}_{(k)} \quad (k \in \mathbf{Z}) \tag{4.12}$$

with $\mathcal{A}_{(k)} \subset \mathcal{A}(\tau)$ in (4.2), we get a decreasing filtration $(H^*_{(k),\,\xi})_k$ of $\xi(H^*) \simeq H^*$ with properties (4.3)-(4.5). Write $\mathrm{Gr}_\xi(H^*)$ for the corresponding $(S(\mathfrak{g}), K)$-module:

$$\bigoplus_k H^*_{(k),\,\xi}/H^*_{(k+1),\,\xi} \subset \mathrm{Gr}\,\mathcal{A}(\tau). \tag{4.13}$$

On the other hand, the filtration $(H^*_{(k),\,\xi})_k$ of H^* gives rise to an increasing filtration $(H_{k,\,\xi})_k$ of H with

$$H_{k,\,\xi} := \{v \in H |\ < w^*, v >= 0 \quad (\forall w^* \in H^*_{(k),\,\xi})\}, \tag{4.14}$$

by passing to the orthogonal in H. If

$$\mathrm{gr}_\xi(H) := \bigoplus_k H_{k+1,\,\xi}/H_{k,\,\xi} \tag{4.15}$$

denotes the corresponding graded $(S(\mathfrak{g}), K)$-module, the dual pairing $< \cdot, \cdot >$ on $H^* \times H$ naturally induces a non-degenerate $(S(\mathfrak{g}), K)$-invariant pairing on $\mathrm{Gr}_\xi(H^*) \times \mathrm{gr}_\xi(H)$. By using the latter pairing, one easily finds that

$$\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Gr}_\xi(H^*) = \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}_\xi(H), \tag{4.16}$$

and that

$$\mathrm{gr}_\xi(H) = \mathrm{gr}(H; H_{0,\xi}) \quad (\text{see (1.1)}). \tag{4.17}$$

We have thus obtained the following proposition, which enables us to describe the associated variety $\mathcal{V}(H)$ of Harish-Chandra module H by means of the annihilator ideal of $\mathrm{Gr}_\xi(H^*)$.

PROPOSITION 4.1. *Under the above notation one has the equality*

$$\mathcal{V}(H) = \{X \in \mathfrak{g} |\ \ f(X) = 0 \ \textit{for all}\ f \in \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{Gr}_\xi(H^*))\}. \tag{4.18}$$

5. Graded modules $\mathrm{Gr}\, H_\Lambda$ and differential operators $\mathcal{D}_\lambda$ of gradient-type.

Let H_Λ be the $(\mathfrak{g}, K)$-module of discrete series π_Λ with Harish-Chandra parameter $\Lambda \in \Xi$. Since the lowest K-type $(\tau_\lambda, V_\lambda)$, $\lambda = \Lambda - \rho_c + \rho_n$, appears in H_Λ with multiplicity one (see Proposition 1.1), there exists a unique, up to scalar multiples, $(\mathfrak{g}, K)$-module embedding ξ_λ from H_Λ into the analytically induced module $\mathcal{A}(\tau_\lambda)$.

This section interprets after Hotta-Parthasarathy [9], the $(S(\mathfrak{g}), K)$-module $\mathrm{Gr}\, H_\Lambda := \mathrm{Gr}_{\xi_\lambda}(H_\Lambda)$ defined in 4.2, by means of the gradient-type differential operator $\mathcal{D}_\lambda$ whose kernel realizes π_Λ. Here we treat H_Λ itself instead of its dual $(\mathfrak{g}, K)$-module H_Λ^*, by noting that

$$H_\Lambda^* \simeq H_{-w_0\Lambda} \quad \text{as } (\mathfrak{g}, K)\text{-modules}, \tag{5.1}$$

for the longest element w_0 of the Weyl group of Δ_c.

5.1. Operator $\mathcal{D}_\lambda$ and realization of discrete series.

Let $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ be dual bases of $\mathfrak{p}$ as in 4.1. We set for $f \in \mathcal{A}(\tau_\lambda)$,

$$\nabla_\lambda f(g) := \sum_{i=1}^{s} R_{X_i} f(g) \otimes X_i^* \quad (g \in G), \tag{5.2}$$

where R_X denotes the left G-invariant vector field on G defined by

$$R_X f(g) := \frac{d}{dt}(f(g \exp tY) + \sqrt{-1} f(g \exp tZ))_{|t=0}$$

for $X = Y + \sqrt{-1}Z$ with $Y, Z \in \mathfrak{g}_0$. It is then easy to see that ∇_λ is independent of the choice of dual bases and that it defines a first order, left G-invariant differential operator from $\mathcal{A}(\tau_\lambda)$ to $\mathcal{A}(\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p})$. Here $\mathrm{Ad}_\mathfrak{p}$ denotes the adjoint representation of K on $\mathfrak{p}$.

Notice that the tensor product K-representation $\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p}$ decomposes into irreducibles as

$$\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p} \simeq \bigoplus_{\beta \in \Delta_n} [m_\beta] \cdot \tau_{\lambda+\beta}, \tag{5.3}$$

where the multiplicity m_β of $\tau_{\lambda+\beta}$ is either 1 or 0 for every $\beta \in \Delta_n$. Let $(\tau_\lambda^\pm, V_\lambda^\pm)$ be the subrepresentations of $\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p}$ such that $\tau_\lambda^\pm \simeq \bigoplus_{\beta \in \Delta_n^+} [m_\beta] \cdot \tau_{\lambda \pm \beta}$, and $P_\lambda : V_\lambda \to V_\lambda^-$ be the projection along the decomposition $V_\lambda = V_\lambda^- \oplus V_\lambda^+$.

We now put

$$\mathcal{D}_\lambda f(g) := P_\lambda(\nabla_\lambda f(g)) \quad (f \in \mathcal{A}(\tau_\lambda)). \tag{5.4}$$

Then $\mathcal{D}_\lambda$ gives a G-invariant differential operator from $\mathcal{A}(\tau_\lambda)$ to $\mathcal{A}(\tau_\lambda^-)$.

It follows immediately from the lowest K-type property of H_Λ that

$$\xi_\lambda(H_\Lambda) \subset \mathrm{Ker}\, \mathcal{D}_\lambda. \tag{5.5}$$

Moreover, the following result, due to Hotta-Parthasarathy, Schmid and Wallach, says that the L^2-kernel of $\mathcal{D}_\lambda$ realizes the discrete series π_Λ.

PROPOSITION 5.1. (Cf. [20, I, Th.1.5]) *For any $\Lambda \in \Xi$, the $(\mathfrak{g}, K)$-module $\xi_\lambda(H_\Lambda)$, isomorphic to H_Λ, consists exactly of all functions $f \in \mathrm{Ker}\mathcal{D}_\lambda$ which are left K-finite and square-integrable on G.*

5.2. Polynomialization $\mathrm{Gr}[\mathcal{D}_\lambda]$ and realization of $\mathrm{Gr}\, H_\Lambda$.

Let $(\mathcal{A}_{(k)})_{k\in\mathbf{Z}}$ (resp. $(\mathcal{A}^-_{(k)})_{k\in\mathbf{Z}}$) be the decreasing filtration of $\mathcal{A}(\tau_\lambda)$ (resp. $\mathcal{A}(\tau_\lambda^-)$), defined by (4.2). Since $\mathcal{D}_\lambda$ sends $\mathcal{A}_{(k)}$ into $\mathcal{A}^-_{(k-1)}$, the operator $\mathcal{D}_\lambda$ induces an $(S(\mathfrak{g}), K)$-homomorphism, say $\mathrm{Gr}[\mathcal{D}_\lambda]$, from $\mathrm{Gr}\,\mathcal{A}(\tau_\lambda)$ to $\mathrm{Gr}\,\mathcal{A}(\tau_\lambda^-)$. Through the isomorphism $\tilde{\iota}$ in Lemma 4.1, we regard this homomorphism as a map

$$\mathrm{Gr}[\mathcal{D}_\lambda]:\ S(\mathfrak{p})\otimes V_\lambda \to S(\mathfrak{p})\otimes V_\lambda^-, \tag{5.6}$$

which is called the *polynomialization of $\mathcal{D}_\lambda$*.

Observe that $\mathrm{Gr}[\mathcal{D}_\lambda]$ is given as

$$(\mathrm{Gr}[\mathcal{D}_\lambda]f)(Y) = P_\lambda(\sum_i (X_i f)(Y)\otimes X_i^*) \quad (Y\in\mathfrak{g}) \tag{5.7}$$

for $f \in S(\mathfrak{p})\otimes V_\lambda$. Here $S(\mathfrak{p})\otimes V$, $V = V_\lambda$ or V_λ^-, is identified in the canonical way with the space of V-valued polynomial functions on $\mathfrak{g}$, vanishing identically on $\mathfrak{k}$.

By virtue of (5.5), one can easily deduce the inclusion

$$\mathrm{Gr}\, H_\Lambda = \mathrm{Gr}_{\xi_\lambda}(H_\Lambda) \subset \mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \tag{5.8}$$

for every Harish-Chandra module H_Λ of discrete series. Furthermore, the discussion in [9, page 160] combined with the Blattner multiplicity formula (cf. [20, I, Prop.1.2]) immediately gives the following theorem.

THEOREM 5.1. (Hotta-Parthasarathy) *The equality $\mathrm{Gr}\, H_\Lambda = \mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ holds in (5.8) provided that the lowest hightest weight $\lambda = \Lambda - \rho_c + \rho_n$ of H_Λ is far from the walls:*

$$\lambda - \sum_{\beta\in Q}\beta \text{ is } \Delta_c^+\text{-dominant for any subset } Q \text{ of } \Delta_n^+. \tag{5.9}$$

Combining this theorem with Proposition 4.1, we make an essential step torward the proof of Theorem 3.1, as in

THEOREM 5.2. *Let H_Λ ($\Lambda \in \Xi$) be a Harish-Chandra module of discrete series, and $H_\Lambda^* \simeq H_{-w_0\Lambda}$ (see (5.1)) be its dual $(\mathfrak{g}, K)$-module. If $\lambda = \Lambda - \rho_c + \rho_n$ is far from the walls, the associated variety $\mathcal{V}(H_\Lambda^*)$ of discrete series H_Λ^* is determined by the annihilator of operator* $\mathrm{Gr}[\mathcal{D}_\lambda]$ *in* (5.6)*:*

$$\mathcal{V}(H_\Lambda^*) = \{X \in \mathfrak{g} |\ f(X) = 0, \quad \forall f \in \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])\}. \tag{5.10}$$

REMARK. By (5.8) and Proposition 4.1, the inclusion $\subset$ is always true in (5.10) without any assumption on the regulality of λ.

6. $(S(\mathfrak{g}), K)$-modules $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ and the corresponding annihilator ideals.

We now go into more detailed structure of graded $(S(\mathfrak{g}), K)$-modules $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \subset S(\mathfrak{p}) \otimes V_\lambda$ defined in 5.2, and their annihilators $\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])) \subset S(\mathfrak{g})$.

6.1. Generating subspaces of $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ as K-modules.

Let $f = X^m \otimes v$ be an element of $S(\mathfrak{p}) \otimes V_\lambda$ with $X \in \mathfrak{p}$, $v \in V_\lambda$ and an integer $m \geq 0$. In view of (5.7) one can compute $\mathrm{Gr}[\mathcal{D}_\lambda]f \in S(\mathfrak{p}) \otimes V_\lambda^-$ as

$$\mathrm{Gr}[\mathcal{D}_\lambda]f = mX^{m-1} \otimes P_\lambda(v \otimes X), \tag{6.1}$$

where P_λ is, as in 5.1, the projection from $V_\lambda = V_\lambda^+ \bigoplus V_\lambda^-$ onto V_λ^-. This implies that f lies in $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ if and only if $v \otimes X \in V_\lambda^+$. Notice that, if v_λ is a non-zero highest weight vector of V_λ, the vector $v_\lambda \otimes X_+$ belongs to V_λ^+ for every $X_+ \in \mathfrak{p}_+ = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha$.

For any subset A of $S(\mathfrak{p}) \otimes V_\lambda$, let $\{A\}_K$ denote the K-submodule of $S(\mathfrak{p}) \otimes V_\lambda$ generated by A. The above discussion leads us to

PROPOSITION 6.1. *The kernel* $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ *contains the K-submodule* $\{S(\mathfrak{p}_+) \otimes v_\lambda\}_K$.

Conversely, we can prove that $\{S(\mathfrak{p}_+) \otimes v_\lambda\}_K$ exhausts $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ in the following sense.

THEOREM 6.1. *For each fixed integer $m \geq 0$, there exists a constant $c_m > 0$ such that*

$$\mathrm{Ker}^m(\mathrm{Gr}[\mathcal{D}_\lambda]) = \{S^m(\mathfrak{p}_+) \otimes v_\lambda\}_K \tag{6.2}$$

holds if the lowest highest weight λ satisfies the condition

$$(\lambda, \alpha) \geq c_m \quad \textit{for all } \alpha \in \Delta_c^+. \tag{6.3}$$

Here $\mathrm{Ker}^m(\mathrm{Gr}[\mathcal{D}_\lambda]) := \mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \cap (S^m(\mathfrak{p}) \otimes V_\lambda)$ *denotes the homogeneous component of* $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ *of degree m.*

This theorem plays a definitive role in proving Theorem 3.1.

To prove Theorem 6.1, we first specify the K-module structure of $\{S^m(\mathfrak{p}_+)\otimes v_\lambda\}_K$. For a positive integer m, let Φ_m denote the totality of all subsets $\delta=\{\beta_1,\beta_2,\dots,\beta_m\}$ consisting of m non-compact roots $\beta_l\in\Delta_n^+$ $(1\le l\le m)$, and we set

$$X(\delta):=X_{\beta_1}\cdots X_{\beta_m}\in S^m(\mathfrak{p}_+), \tag{6.4}$$

$$\gamma(\delta):=\beta_1+\cdots+\beta_m\in\mathfrak{t}^*. \tag{6.5}$$

Here $X_\beta\in\mathfrak{g}_\beta$ $(\beta\in\Delta_n^+)$ are fixed non-zero root vectors. Note that $X(\delta)$ $(\delta\in\Phi_m)$ form a basis of the linear space $S^m(\mathfrak{p}_+)$.

Let $\{\alpha_1,\dots,\alpha_u\}$ be the simple system of Δ_c^+. Define a positive number κ_m as the maximum of coefficients $n_k(\delta,\delta')$ $(1\le k\le u)$ of expansions:

$$\gamma(\delta)-\gamma(\delta')=\sum_{k=1}^{u}n_k(\delta,\delta')\alpha_k,$$

where δ and δ' range over the elements of Φ_m such that $\gamma(\delta)-\gamma(\delta')\in\sum_{1\le k\le u}\mathbf{R}\alpha_k$.

LEMMA 6.1. *For a positive integer m, one has an isomorphism of K-modules*

$$\{S^m(\mathfrak{p}_+)\otimes v_\lambda\}_K\simeq\bigoplus_{\delta\in\Phi_m}(\tau_{\lambda+\gamma(\delta)},V_{\lambda+\gamma(\delta)}) \tag{6.6}$$

if the lowest highest weight λ fulfills the condition:

$$\frac{2(\lambda,\alpha_k)}{(\alpha_k,\alpha_k)}>\kappa_m \tag{6.7}$$

for every simple root α_k $(1\le k\le u)$ of Δ_c^+.

PROOF. We introduce on the real vector space $\sqrt{-1}\mathfrak{t}_0^*=\sum_{\beta\in\Delta}\mathbf{R}\beta$ a lexicographic order $>$ for which $\Delta^+=\{\beta\in\Delta|\ \beta>0\}$, and arrange the elements of Φ_m as $\delta_0,\delta_1,\dots,\delta_r$ so that $\gamma(\delta_0)\ge\gamma(\delta_1)\ge\cdots\ge\gamma(\delta_r)$. Let $\mathfrak{b}:=\mathfrak{t}+\sum_{\alpha\in\Delta_c^+}\mathfrak{g}_\alpha$ be a Borel subalgebra of $\mathfrak{k}$. Note that the subspace $S^m(\mathfrak{p}_+)\otimes v_\lambda\subset S^m(\mathfrak{p})\otimes V_\lambda$ is $\mathfrak{b}$-stable. Setting

$$A_j:=\sum_{0\le i\le j}\mathbf{C}X(\delta_i)\otimes v_\lambda\subset S^m(\mathfrak{p}_+)\otimes v_\lambda \tag{6.8}$$

for $0\le j\le r$, we get a $\mathfrak{b}$-stable flag $\{A_j\}_j$ of $A_r=S^m(\mathfrak{p}_+)\otimes v_\lambda$.

Let $\{A_j\}_K$ be the K-submodule of $S^m(\mathfrak{p})\otimes V_\lambda$ generated by subspace A_j. One sees that $\{A_j\}_K=U(\mathfrak{n}_-)A_j$ with $\mathfrak{n}_-:=\sum_{\alpha\in\Delta_c^+}\mathfrak{g}_{-\alpha}$, by bearing in mind the decomposition $\mathfrak{k}=\mathfrak{b}+\mathfrak{n}_-$. Moreover, if $X(\delta_j)\otimes v_\lambda$ is not contained in $\{A_{j-1}\}_K$, this vector naturally

gives rise to a non-zero highest vector of quotient K-module $\{A_j\}_K/\{A_{j-1}\}_K$, and one has

$$\{A_j\}_K/\{A_{j-1}\}_K \simeq (\tau_{\lambda+\gamma(\delta_j)}, V_{\lambda+\gamma(\delta_j)}) \tag{6.9}$$

as K-modules. So, to complete the proof, it is enough to show that

$$X(\delta_j) \otimes v_\lambda \notin \{A_{j-1}\}_K = U(\mathfrak{n}_-)A_{j-1}$$

for every j under the assumption (6.7) on λ.

Suppose by contraries that $X(\delta_j) \otimes v_\lambda \in U(\mathfrak{n}_-)A_{j-1}$ for some j. Then there exist elements $D_i \in U(\mathfrak{n}_-)$ $(i < j)$ such that

$$X(\delta_j) \otimes v_\lambda = \sum_{i<j} D_i(X(\delta_i) \otimes v_\lambda), \tag{6.10}$$

and that

$$[Z, D_i] = (\gamma(\delta_j) - \gamma(\delta_i))(Z)D_i \quad (Z \in \mathfrak{t}). \tag{6.11}$$

If i_0 is the smallest number i with $D_i \neq 0$, the right hand side of (6.10) turns out to be

$$X(\delta_{i_0}) \otimes D_{i_0}v_\lambda \mod S^m(\mathfrak{p}) \otimes \{\sum_{\mu>\lambda+\gamma(\delta_j)-\gamma(\delta_{i_0})} V_\lambda(\mu)\}, \tag{6.12}$$

where $V_\lambda(\mu)$ is the weight space of V_λ of weight μ. We thus find from (6.10) that $D_{i_0}v_\lambda = 0$.

It then follows from the celebrated Bernstein-Gelfand-Gelfand resolution of V_λ in terms of Verma $U(\mathfrak{k})$-modules (cf. [14, Lemma 2.2.10]) that the weight $\gamma(\delta_j) - \gamma(\delta_{i_0})$ of D_{i_0} must be of the form

$$-\frac{2(\lambda, \alpha_l)}{(\alpha_l, \alpha_l)}\alpha_l - \sum_{1\leq k\leq u} n_k\alpha_k \quad (n_k \geq 0, \in \mathbf{Z}) \tag{6.13}$$

for some simple root $\alpha_l \in \Delta_c^+$. This contradicts to the assumption (6.7). Q.E.D.

Now let K^C be the complexification of compact group K, and B be the Borel subgroup of K^C with Lie algebra $\mathfrak{b}$. For a holomorphic B-module F, we write $H^i(F)$ $(i = 0, 1, \ldots)$ for the i-th cohomology group of K^C/B with coefficients in the sheaf of germs of holomorphic sections of vector bundle $K^C \times_B F$ (see e.g., [17, 3.1.2] for the definition of $H^i(F)$). Then $H^i(F)$ admits a natural K^C-module structure.

For a T-integrable linear form μ on $\mathfrak{t}_0 = \mathrm{Lie}(T)$, let $C(\mu)$ denote the one-dimensional B-module with differential μ extended to $\mathfrak{b}$ trivially on the nilradical $\mathfrak{n} = \sum_{\alpha\in\Delta_c^+} \mathfrak{g}_\alpha$.

The following lemma, due to Hotta-Parthasarathy, describes the K-module in the left hand side of (6.2) by means of cohomology group of K^C/B.

LEMMA 6.2. ([9, Lemma 5.2]) *The K-module* $\mathrm{Ker}^m(\mathrm{Gr}[\mathcal{D}_\lambda])$ *is isomorphic to the cohomology group* $H^t(S^m(\mathfrak{p}_+)\otimes C(\lambda+2\rho_c))$ $(t:=\dim K^C/B)$ *for every integer* $m\geq 0$, *provided that the parameter* λ *is far from the walls in the sense of* (5.9).

The above two lemmas enable us to prove Theorem 6.1, as follows.

PROOF OF THEOREM 6.1. Fix a non-negative integer m, and set

$$\tilde{A}_j := \sum_{i\leq j} \mathbf{C}X(\delta_j)\otimes C(\lambda+2\rho_c) \subset S^m(\mathfrak{p}_+)\otimes C(\lambda+2\rho_c) \quad (0\leq j\leq r),$$

where $\{X(\delta_j)\}_{0\leq j\leq r}$ is the basis of $S^m(\mathfrak{p}_+)$ defined in (6.4). Then one has a flag of B-modules

$$(6.14)\qquad (0)\subset \tilde{A}_0\subset \tilde{A}_1\subset\cdots\subset\tilde{A}_{j-1}\subset\tilde{A}_j\subset\cdots\subset\tilde{A}_r = S^m(\mathfrak{p}_+)\otimes C(\lambda+2\rho_c)$$

such that

$$(6.15)\qquad 0\to\tilde{A}_{j-1}\to\tilde{A}_j\to C(\lambda+\gamma(\delta_j)+2\rho_c)\to 0 \quad \text{as } B\text{-modules.}$$

Now assume that λ satisfies the regurality condition (6.7) in Lemma 6.1. Then one finds that the linear form $\lambda+\gamma(\delta_j)$ is Δ_c^+-dominant for every j. It follows from the Borel-Weil-Bott theorem (see e.g., [17, Th.3.1.2.2]) that

$$(6.16)\qquad H^p(C(\lambda+\gamma(\delta_j)+2\rho_c))\simeq\begin{cases} V_{\lambda+\gamma(\delta_j)} & (p=t),\\ (0) & (p\neq t),\end{cases}$$

as K^C-modules. Hence (6.15) induces an exact sequence of K^C-modules

$$(6.17)\qquad 0\to H^t(\tilde{A}_{j-1})\to H^t(\tilde{A}_j)\to V_{\lambda+\gamma(\delta_j)}\to 0,$$

and consequently we deduce

$$(6.18)\qquad H^t(S^m(\mathfrak{p}_+)\otimes C(\lambda+2\rho_c))\simeq\bigoplus_{\delta\in\Phi_m}(\tau_{\lambda+\gamma(\delta)},V_{\lambda+\gamma(\delta)}).$$

This together with Proposition 6.1, Lemmas 6.1 and 6.2 completes the proof of Theorem 6.1. Q.E.D.

6.2. Annihilator ideal $\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$.

For a subset E of $\mathfrak{g}$, let $\mathcal{I}(E)$ denote the ideal of $S(\mathfrak{g})$ determined by E:

$$(6.19)\qquad \mathcal{I}(E):=\{f\in S(\mathfrak{g})|\ f(X)=0\ \ (\forall X\in E)\}.$$

Two results in 6.1 allow us to establish the following

THEOREM 6.2. *Let $\lambda = \Lambda - \rho_c + \rho_n$ be the lowest highest weight of discrete series H_Λ. Then one has*

$$\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \subset \mathcal{I}(K_C\mathfrak{p}_+). \tag{6.20}$$

Moreover there exists a positive constant c such that the equality holds in (6.20) *provided that $(\lambda, \alpha) \geq c$ for all $\alpha \in \Delta_c^+$.*

This theorem together with Theorem 5.1 immediately yields

COROLLARY 6.1. *If the lowest highest weight λ is sufficiently Δ_c^+-regular, the annihilator ideal of graded $S(\mathfrak{g})$-module* $\mathrm{Gr}\, H_\Lambda$ *(see* 5.1*) coincides with its radical.*

PROOF OF THEOREM 6.2. The inclusion (6.20) follows immediately from Proposition 6.1. To prove the second assertion, note at first that $\mathcal{I}(K_C\mathfrak{p}_+)$ is a graded ideal of $S(\mathfrak{g})$ containing $\mathfrak{k}S(\mathfrak{g})$. Since $S(\mathfrak{g})$ is a Noetherian ring, there exist a finite number of homogeneous elements $D_j \in S(\mathfrak{p})$ $(1 \leq j \leq r)$ such that

$$\mathcal{I}(K_C\mathfrak{p}_+) = \mathfrak{k}S(\mathfrak{g}) + S(\mathfrak{g})D_1 + \cdots + S(\mathfrak{g})D_r.$$

Let c_j be the positive constants in Theorem 6.1 associated to $d_j := \deg D_j$ $(1 \leq j \leq r)$, and put $c := \max_j(c_j)$. Then (6.2) tells us that, if $(\lambda, \alpha) \geq c$ $(\forall \alpha \in \Delta_c^+)$, then each D_j is identically zero on $\mathrm{Ker}^{d_j}(\mathrm{Gr}[\mathcal{D}_\lambda])$. One easily sees from this fact that D_j annihilates all the vectors in $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$. We thus conclude $\mathcal{I}(K_C\mathfrak{p}_+) = \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ as desired. Q.E.D.

7. Completion of the proof of Theorem 3.1.

By virtue of Theorems 5.2 and 6.2, we find that

$$\mathcal{V}(H_\Lambda^*) = K_C\mathfrak{p}_+, \tag{7.1}$$

if the corresponding lowest highest weight λ is sufficiently Δ_c^+-regular. Here it should be noticed that $K_C\mathfrak{p}_+$ is (Zariski) closed in $\mathfrak{p}$, because $\mathfrak{p}_+$ is stable under the Borel subgroup B while the flag variety K_C/B is complete. A standard argument of Zuckerman's translation principle, given below, allows us to show that (7.1) is always true for any $\Lambda \in \Xi$. This together with (5.1) will complete the proof of Theorem 3.1.

To be more precise, we use the following proposition.

PROPOSITION 7.1. (Cf. [2, Prop.1.3]) *Let H be a finitely generated $U(\mathfrak{g})$-module, and L be a $U(\mathfrak{g})$-submodule of H. Then L is finitely generated over $U(\mathfrak{g})$, and its associated variety $\mathcal{V}(L)$ is contained in $\mathcal{V}(H)$.*

PROOF. Take a finite-dimensional subspace H_0 of H such that $H = U(\mathfrak{g})H_0$, and set $H_k := U_k(\mathfrak{g})H_0$, $L_k := H_k \cap L$ for each integer $k \geq 0$. Then $(L_k)_{k=0,1,\ldots}$ gives an increasing filtration of $U(\mathfrak{g})$-module L such that $U_m(\mathfrak{g})L_k \subset L_{m+k}$, and correspondingly one has graded $S(\mathfrak{g})$-modules

$$\mathrm{gr}(H; H_0) = \bigoplus_k M_k \text{ (see (1.1))} \quad \text{and} \quad \mathrm{gr}(L) = \bigoplus_k N_k$$

with $M_k := H_k/H_{k-1}$ and $N_k := L_k/L_{k-1}$ respectively. Note that the latter $\mathrm{gr}(L)$ can be looked upon as an $S(\mathfrak{g})$-submodule of the former $\mathrm{gr}(H; H_0)$ in the canonical way. Since the ring $S(\mathfrak{g})$ is Noetherian and since $\mathrm{gr}(H; H_0)$ is of finite type over $S(\mathfrak{g})$, we find that $\mathrm{gr}(L)$ also is finitely generated as an $S(\mathfrak{g})$-module. Hence there exists an integer $j \geq 0$ such that $\mathrm{gr}(L) = S(\mathfrak{g})(N_0 + \cdots + N_j)$ and so we obtain $L = U(\mathfrak{g})L_j$. This shows that L is finitely generated over $U(\mathfrak{g})$.

The second claim $\mathcal{V}(L) \subset \mathcal{V}(H)$ can be shown just as in the proof of [21, Th.2.2]. Actually, for the above j we can prove

$$N_k = S^{k-j}(\mathfrak{g})N_j, \quad L_k = U_{k-j}(\mathfrak{g})L_j \quad (k = j, j+1, \ldots), \tag{7.2}$$

which imply that

$$\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}(L)} = \sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}(L; L_j)}. \tag{7.3}$$

Here $\mathrm{gr}(L; L_j)$ is the graded $S(\mathfrak{g})$-module defined through the filtration $(U_k(\mathfrak{g})L_j)_k$ of L, and $\sqrt{J}$ denotes the radical of an ideal J of $S(\mathfrak{g})$. The inclusion $\mathcal{V}(L) \subset \mathcal{V}(H)$ now follows by taking the varieties associated to the ideals in (7.3), since $\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}(L)}$ includes the ideal $\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}(H; H_0)}$ which defines the variety $\mathcal{V}(H)$. Q.E.D.

We are now in a position to complete the proof of Theorem 3.1.

In view of (5.1) it sufficies to show (7.1) for $\Lambda = \lambda + \rho_c - \rho_n \in \Xi$ when the lowest highest weight λ is *not* sufficiently Δ_c^+-regular. For such a Λ, we can take an irreducible finite-dimensional $(\mathfrak{g}, K)$-module Y_μ with Δ^+-highest weight μ for which the linear form $\lambda + \mu$ is sufficiently Δ_c^+-regular and so (7.1) holds for $\Lambda + \mu$.

By making use of Zuckerman's translation fanctors (see e.g., [20, I, 3.4]), we deduce that $H^*_{\Lambda+\mu}$ (resp. H^*_Λ) is isomorphic to a $(\mathfrak{g}, K)$-submodule of the tensor product $H^*_\Lambda \otimes Y^*_\mu$ (resp. $H^*_{\Lambda+\mu} \otimes Y_\mu$). It then follows from Proposition 7.1 together with [3, Lemma 4.1] that

$$\begin{aligned} \mathcal{V}(H^*_{\Lambda+\mu}) &\subset \mathcal{V}(H^*_\Lambda \otimes Y^*_\mu) = \mathcal{V}(H^*_\Lambda) \\ &\subset \mathcal{V}(H^*_{\Lambda+\mu} \otimes Y_\mu) = \mathcal{V}(H^*_{\Lambda+\mu}). \end{aligned}$$

We thus conclude from (7.1) applied to $\Lambda + \mu$, that $\mathcal{V}(H^*_\Lambda) = K_C\mathfrak{p}_+$ as desired. Now Theorem 3.1 is completely proved.

8. Nilpotent orbits $\mathcal{O}_{\mathfrak{p}_-}$ and explicit formulae for the Gelfand-Kirillov dimensions $d(H_\Lambda)$.

We finish this article with giving explicit formulae for the Gelfand-Kirillov dimensions $d(H_\Lambda) = \dim \mathcal{V}(H_\Lambda)$ of discrete series, by specifying the nilpotent orbits $\mathcal{O}_{\mathfrak{p}_-}$ in Proposition 3.2. We concentrate here on the case of unitary groups $G = SU(p,q)$ with integers $p, q \geq 0$, $(p,q) \neq (0,0)$. Our formula for $d(H_\Lambda)$ in Theorem 8.3 is recursive with respect to the parameter $n := p + q$.

8.1. The mappings GKD and OBT.

Realize our group G as

$$G = \{g \in SL(n, \mathbf{C}) |\ {}^t\overline{g} I_{p,q} g = I_{p,q}\} \quad (n = p+q) \tag{8.1}$$

with

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ O & -I_q \end{pmatrix} \quad (I_r \text{ the identitiy matrix of degree } r),$$

where ${}^t g$ (resp. $\overline{g}$) denotes the transposed (resp. the complex conjugate) of a matrix g. Then the Lie algebras $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ and subspace $\mathfrak{p}$ can be written as follows:

$$\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C}) := \{X \in M(n,n) |\ \mathrm{tr}\, X = 0\}, \tag{8.2}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \in \mathfrak{g} \;\middle|\; Y_1 \in M(p,p),\ Y_2 \in M(q,q) \right\}, \tag{8.3}$$

$$\mathfrak{t} = \{Z = \mathrm{diag}(t_1, ..., t_n) \mid t_i \in \mathbf{C},\ \mathrm{tr}\, Z = 0\}, \tag{8.4}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix} \in \mathfrak{g} \;\middle|\; V \in M(p,q),\ W \in M(q,p) \right\}. \tag{8.5}$$

Here $M(p,q)$ denotes the space of complex matrices of size $p \times q$. The root system Δ (resp. $\Delta_c \subset \Delta$) of $\mathfrak{g}$ (resp. $\mathfrak{k}$) with respect to $\mathfrak{t}$ is of type A_{n-1} (resp. $A_{p-1} \times A_{q-1}$), and it is described respectively as

$$\Delta = \{e_{ij} |\ 1 \leq i, j \leq n, i \neq j\}, \quad \Delta_c = \{e_{ij} \in \Delta |\ 1 \leq i, j \leq p \ \text{ or } \ p < i, j \leq n\} \tag{8.6}$$

with $e_{ij}(Z) := t_i - t_j$ $(Z \in \mathfrak{t})$. We fix as in §2 a positive system of Δ_c:

$$\Delta_c^+ := \{e_{ij} \in \Delta_c |\ i < j\}. \tag{8.7}$$

Let $\Pi_{p,q}$ be the totality of maps h from $F(n) := \{1, 2, \ldots, n\}$ to the set $\{a, b\}$ of two elements a and b, such that

$$\#(h^{-1}(\{a\})) = p, \quad \text{and} \quad \#(h^{-1}(\{b\})) = q,$$

where $\#(S)$ denotes the cardinal number of a set S. For an $h \in \Pi_{p,q}$, arrange the elements of $h^{-1}(\{a\})$ and $h^{-1}(\{b\})$ respectively as

$$(w(1), w(2), \ldots, w(p)) \quad \text{with} \quad w(1) < w(2) < \ldots < w(p),$$

$$(w(p+1), w(p+2), \ldots, w(n)) \quad \text{with} \quad w(p+1) < w(p+2) < \ldots < w(n),$$

and we put

$$\Delta^+(h) := \{e_{ij} \in \Delta |\ w(i) < w(j)\} \tag{8.8}$$

through this permutation w of $F(n)$.

It is then immediate to verify

LEMMA 8.1. *The assignment $h \to \Delta^+(h)$ gives a bijective correspondence from $\Pi_{p,q}$ onto the totality of positive systems of Δ including Δ_c^+.*

Now let H_Λ be the discrete series module with Harish-Chandra parameter $\Lambda \in \Xi$. This parameter set Ξ is written as a disjoint union of subsets $\Xi(h) := \{\Lambda \in \Xi |\ \Lambda \text{ is } \Delta^+(h)\text{-dominant}\}$ $(h \in \Pi_{p,q})$. Noting that, for each fixed h, the Gelfand-Kirillov dimension $d(H_\Lambda)$ and nilpotent $\mathrm{Ad}(G^C)$-orbit $\mathcal{O}_{\mathfrak{p}_-}$ in Proposition 3.2 are independent of the choice of a $\Lambda \in \Xi(h)$, one can define well-defined mappings:

$$\mathrm{GKD}_{p,q} :\ \Pi_{p,q} \ni h \to d(H_\Lambda) \in \{0, 1, 2, ...\}, \tag{8.9}$$

and

$$\mathrm{OBT}_{p,q} :\ \Pi_{p,q} \ni h \to \mathcal{O}_{\mathfrak{p}_-} \in \{\text{nilpotent } \mathrm{Ad}(G^C)\text{-orbits in } \mathfrak{g}\}, \tag{8.10}$$

where $G^C = SL(n, \mathbf{C})$. We call $\mathrm{GKD}_{p,q}$ (resp. $\mathrm{OBT}_{p,q}$) the *Gelfand-Kirillov dimension map* (resp. *nilpotent orbit map*) associated with the discrete series of $G = SU(p,q)$. It follows from Proposition 3.2 that

$$\mathrm{GKD}_{p,q}(h) = \frac{1}{2} \dim \mathrm{OBT}_{p,q}(h) \quad (h \in \Pi_{p,q}). \tag{8.11}$$

Put $\Pi := \cup_{p,q} \Pi_{p,q}$ (disjoint union) by varying the non-negative integers p and q. Then $\mathrm{GKD}_{p,q}$ and $\mathrm{OBT}_{p,q}$ extend naturally to mappings defined on Π which we denote respectively by

$$\mathrm{GKD} = \bigoplus_{p,q} \mathrm{GKD}_{p,q}, \quad \mathrm{OBT} = \bigoplus_{p,q} \mathrm{OBT}_{p,q}. \tag{8.12}$$

It should be noticed that, for an integer $n > 0$, the subset $\Pi(n) := \cup_{p+q=n} \Pi_{p,q} \subset \Pi$ consists of all mappings from $F(n)$ to $\{a, b\}$.

8.2. Specification of OBT(h) and explicit formulae for GKD(h).

We now introduce an assignment $R : \Pi \to \Pi$ which allows us to describe the mappings GKD and OBT explicitly.

Let h be in $\Pi(n)$ with an integer $n > 0$. We say that two elements $i, j \in F(n)$ are connected with respect to h, or $i \sim j$ for short, if the function h is constant on the segment $[i, j] \subset F(n)$. This $\sim$ clearly gives an equivalence relation on $F(n)$. Each equivalence class of $(F(n), \sim)$, viewed as a subset of $F(n)$, is called an *h-connected component* of $F(n)$.

Take a complete system $I \subset F(n)$ of representatives of the set of h-connected components, and let ζ be the unique bijection

$$\zeta : \ F(n) \setminus I \to F(n - |h|), \tag{8.13}$$

characterized by the condition

$$i < j \Leftrightarrow \zeta(i) < \zeta(j) \qquad \text{for } \ i, j \in F(n) \setminus I.$$

Here $|h|$ denotes the number of h-connected components.

We define $Rh \in \Pi(n - |h|)$ by

$$Rh := h \circ \zeta^{-1}. \tag{8.14}$$

Note that Rh is independent of the choice of a set of representatives I. Since $\Pi = \cup_{n>0}\Pi(n)$ (disjoint union), $R : \Pi(n) \to \Pi(n - |h|)$ naturally extends to an assingment defined on Π, which we denote by the same letter R.

Applying R repeatedly, we obtain from each $h \in \Pi(n)$ a finite sequence $(R^k(h))_{0 \le k \le l}$ of elements of Π with

$$R^k(h) \in \Pi(n_k(h)), \quad n_k(h) = n - \sum_{j=0}^{k-1} |R^j(h)|. \tag{8.15}$$

Here l is the non-negative integer such that

$$|R^l(h)| = n_l(h) > 0. \tag{8.16}$$

Notice that $(r_k)_{0 \le k \le l}$ with $r_k := |R^k(h)|$, gives a partition of n. We put

$$J[h] := J(r_0) \oplus J(r_1) \oplus \cdots \oplus J(r_l) \in \mathfrak{g}, \tag{8.17}$$

where $J(m)$ denotes the Jordan matrix of degree m given by

$$J(m) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}, \quad \text{and} \quad A \oplus B := \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

The following theorem specifies the nilpotent $\mathrm{Ad}(G^C)$-orbits associated with the discrete series.

THEOREM 8.1. *One has* $\mathrm{OBT}(h) = \mathrm{Ad}(G^C) \cdot J[h]$ *for every* $h \in \Pi(n)$, *where the nilpotent class* $\mathrm{OBT}(h)$ *is defined by* (8.10) *and* (8.12).

We will prove this theorem in the last subsection 8.3.

The above theorem together with (8.11) enables us to give an explicit formula for the GKD, as follows.

THEOREM 8.2. *The Gelfand-Kirillov dimension of an* $h \in \Pi(n) = \cup_{p+q=n}\Pi_{p,q}$ $(n > 0)$ *is given as*

$$\mathrm{GKD}(h) = \frac{1}{2}\sum_{k=0}^{l}(2n_k(h) - r_k)(r_k - 1) = \frac{1}{2}(n^2 - \sum_{k=0}^{l}(2k+1)r_k) \tag{8.18}$$

with $r_k = |R^k(h)|$, *by means of the finite sequences* $(R^k(h))_{0\le k\le l}$ *and* $(n_k(h))_{0\le k\le l}$ *in* (8.15) *and* (8.16).

The second equality in (8.18) follows by a direct computation. The first one is rewritten immediately to the following recursion formula, which is easier to handle.

THEOREM 8.3. *It holds that*

$$\mathrm{GKD}(h) = \mathrm{GKD}(Rh) + (2n - |h|)(|h| - 1)/2 \tag{8.19}$$

for $h \in \Pi(n)$, *where we set* $\mathrm{GKD}(Rh) = 0$ *for* h*'s such that* $|h| = n$.

REMARK. An $h \in \Pi(n)$ satisfies the condition $|h| = n$ if and only if the Gelfand-Kirillov dimension $d(H_\Lambda)$ of corresponding discrete series equals $\#(\Delta_+) = n(n-1)/2$, i.e., H_Λ is large in the sense of [15, §6].

REMARK. Joseph [10, Th.4.1] gave a combinatorial method for computing the Gelfand-Kirillov dimensions $\dim \mathcal{V}(U(\mathfrak{g})/I)$ of primitive quotients $U(\mathfrak{g})/I$ of $U(\mathfrak{g})$, associated to the annihilator ideals I of simple highest weight $U(\mathfrak{g})$-modules. A fundamental result of Duflo [4] says that the above I's exhaust all primitive ideals of $U(\mathfrak{g})$. Accordingly, our formula (8.18) for $\mathrm{GKD}(h) = (1/2) \cdot \dim \mathcal{V}(U(\mathfrak{g})/I_\Lambda)$ $(\Lambda \in \Xi(h))$ can be related to Joseph's one if one could specify explicitly for each Λ, a simple highest weight $U(\mathfrak{g})$-module whose kernel equals I_Λ. Here I_Λ denotes, as in 3.1, the annihilator ideal of discrete series $U(\mathfrak{g})$-module H_Λ.

PROOF OF THEOREM 8.3. Let $(\hat{r}_i)_{0\le i\le m}$, $m = r_0 - 1$, be the dual partition of $(r_k)_{0\le k\le l}$, $r_k = |R^k(h)|$, i.e., $\hat{r}_i := \#\{k|\ r_k \ge i+1\}$. It then follows from Theorem 8.1 and [12, Prop.1.3] that

$$\dim \mathrm{OBT}(h) = n^2 - \sum_{i=0}^{m} \hat{r}_i^2, \tag{8.20}$$

and that

$$\dim \mathrm{OBT}(Rh) = (n-r_0)^2 - \sum_{i=0}^{m} (\hat{r}_i - 1)^2, \tag{8.21}$$

where $\dim \mathrm{OBT}(Rh) = 0$ if and only if $n = r_0 = |h|$. In view of (8.11), we deduce

$$\begin{aligned} \mathrm{GKD}(h) - \mathrm{GKD}(Rh) &= (\dim \mathrm{OBT}(h) - \dim \mathrm{OBT}(Rh))/2 \\ &= (2n - r_0)(r_0 - 1)/2, \end{aligned}$$

as desired. Q.E.D.

EXAMPLE: CASE OF $G = SU(p,2)$ $(p \ge 2)$. In this case, the set $\Pi_{p,2}$ is divided onto 7 subfamilies according to the positions of two elements $i_1, i_2 \in F(n)$ such that $h(i_1) = h(i_2) = b$, and the corresponding quantities $(r_k)_{0\le k\le l}$ and $\mathrm{GKD}(h)$ are given explicitly as follows.

type	$(h(i))_i$	$(r_k)_k$	$\mathrm{GKD}(h)$
I	$(bba\dots a)$	$(2,2,1,\dots,1)$	$2p$
II	$(ba\dots aba\dots a)$	$(4,1,\dots,1)$	$3p$
III	$(ba\dots ab)$	$(3,1,\dots,1)$	$2p+1$
IV	$(a\dots abba\dots a)$	$(3,3,1,\dots,1)$ $(p\ge 4)$	$4p-4$
		$(3,2)$ $(p=3)$	8
		$(3,1)$ $(p=2)$	5
V	$(a\dots aba\dots aba\dots a)$ $(p\ge 3)$	$(5,1,\dots,1)$	$4p-2$
VI	$(a\dots aba\dots ab)$	$(4,1,\dots,1)$	$3p$
VII	$(a\dots abb)$	$(2,2,\dots,1)$	$2p$

Here two extreme types I and VII correspond to the holomorphic and anti-holomorphic discrete series.

8.3. Proof of Theorem 8.1.

Let h be in $\Pi_{p,q} \subset \Pi(n)$ with $n = p+q > 0$, and $J[h] = \bigoplus_{0\le k\le l} J(r_k)$, $r_k = |R^k(h)|$, be the nilpotent matrix in (8.17). We first show the following

LEMMA 8.2. *The nilpotent orbit* $\mathrm{Ad}(G^C)\cdot J[h]$ *intersects the subspace* $\mathfrak{p}_-$*, i.e.,* $\mathrm{Ad}(G^C)\cdot J[h] \in \Omega_{\mathfrak{p}_-}$ *in the sense of Proposition 3.2.*

PROOF. For each $0 \leq k \leq l$, let $I_k = \{i_1^k, \ldots, i_{r_k}^k\} \subset F(n_k(h))$ be a complete system of representatives of the set of $R^k(h)$-connected components of $F(n_k(h))$, arranged as $i_1^k < \cdots < i_{r_k}^k$. Define a bijective map $\zeta_k : F(n_k(h)) \setminus I_k \to F(n_{k+1}(h))$ as in (8.13), and let j_s^k $(1 \leq s \leq r_k)$ be the element of $F(n)$ such that

$$w(j_s^k) = (\zeta_{k-1} \cdots \zeta_0)^{-1}(i_s^k), \tag{8.22}$$

where $w : F(n) \to F(n)$ is as in (8.8). Then,

$$F(n) = \amalg_{0 \leq k \leq l}\{j_1^k, \ldots, j_{r_k}^k\}, \tag{8.23}$$

gives a partition of $F(n)$.

We now define a matrix $E[h] \in \mathfrak{g}$ by

$$E[h] := E_0 \oplus \cdots \oplus E_l \tag{8.24}$$

with

$$E_k := E(j_{r_k}^k, j_{r_k-1}^k) + \cdots + E(j_2^k, j_1^k). \tag{8.25}$$

Here $E(i,j)$ denotes a root vector corresponding to $e_{ij} \in \Delta$, whose (m,m')-element is 1 if $(m,m') = (i,j)$; and it is 0 otherwise. Clearly, $E[h]$ is conjugate to $J[h]$ under the adjoint action of $G^C = SL(n, \mathbf{C})$.

Notice that $w(j_s^k) < w(j_{s+1}^k)$ and that $(R^k(h))(i_s^k) \neq (R^k(h))(i_{s+1}^k)$ for every s and k. The former is by (8.22) and the latter immediately follows from the definition of I_k. These two facts imply that $E(j_{s+1}^k, j_s^k) \in \mathfrak{p}_-$. We thus find $E[h] \in \mathfrak{p}_- \cap \mathrm{Ad}(G^C) \cdot J[h]$. This proves the lemma. Q.E.D.

Secondly, let $\gamma = (\gamma_0, \ldots, \gamma_m)$ be a finite sequence of positive integers such that $\gamma_0 + \cdots + \gamma_m = n$. Let $\mathfrak{u}(\gamma)$ be the Lie subalgebra of $\mathfrak{g}$ consisting of nilpotent matrices of the form:

$$\begin{pmatrix} O_{00} & O_{01} & \ldots & O_{0m} \\ * & O_{11} & \ldots & O_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & O_{mm} \end{pmatrix},$$

where O_{ij} denotes the zero matrix of size $i \times j$. Note that $\mathfrak{u}(\gamma)$ is the nilradical of a standard papabolic subalgebra of $\mathfrak{g}$.

In order to show that our orbit $\mathrm{Ad}(G^C) \cdot J[h]$ is maximal in $\Omega_{\mathfrak{p}_-}$ with respect to the closure relation, we essentially make use of the following proposition, which is a direct consequence of [8, §4].

PROPOSITION 8.1. *$\mathrm{Ad}(G^C)\cdot J[h]$ is the unique nilpotent G^C-orbit $\mathcal{O}_1$ in $\mathfrak{g}$ with the following property: $\mathcal{O}_1\cap\mathfrak{u}(\gamma)$ is dense in $\mathfrak{u}(\gamma)$ for any finite sequence $\gamma=(\gamma_0,\ldots,\gamma_m)$ gained by a rearrangement of the dual partition $(\hat{r}_i)_{0\le i\le m}$ of $(r_k)_{0\le k\le l}$, $r_k=|R^k(h)|$.*

REMARK. The above $\mathcal{O}_1=\mathrm{Ad}(G^C)\cdot J[h]$ is called the *Richardson orbit* associated to $\mathfrak{u}(\gamma)$.

We are now ready to finish the proof of Theorem 8.1.

Let $C_0,C_1,\ldots,C_{m'}$ be the h-connected components of $F(n)$, arranged as

$$q_1\in C_{i_1},\ q_2\in C_{i_2}\ \text{ and }\ i_1<i_2\quad\Rightarrow\quad q_1<q_2.$$

Set $\gamma_i:=\#(C_i)$. It is then easy to see that $m=m'$ and that $\gamma:=(\gamma_i)_{0\le i\le m}$ is a rearrangement of the dual partition $(\hat{r}_i)_{0\le i\le m}$. Noting that the set $-(\Delta^+(h)\cap\Delta_n)$ of non-compact negative roots is written as

$$\{e_{w^{-1}(i)w^{-1}(j)}\mid\ 1\le w^{-1}(i)\le p,\ \ p<w^{-1}(j)\le n,\ \ i>j\}$$

by means of the permutation w of $F(n)$ in 8.1, one deduces

$$\tilde{w}\mathfrak{p}_-\tilde{w}^{-1}\subset\mathfrak{u}(\gamma). \tag{8.26}$$

Here $\tilde{w}\in G^C$ is a matrix of the form $\sum_{1\le k\le n}\epsilon_k E(w(k),k)$ with $\epsilon_k=1$ or -1, representing the permutation w.

Now, if $\mathcal{O}'$ is a nilpotent $\mathrm{Ad}(G^C)$-orbit in $\mathfrak{g}$ such that $\mathcal{O}'\cap\mathfrak{p}_-\neq\emptyset$, then one finds $\mathfrak{u}(\gamma)\cap\mathcal{O}'\neq\emptyset$ by (8.26). Proposition 8.1 allows us to conclude $\mathcal{O}'\subset\overline{\mathrm{Ad}(G^C)\cdot J[h]}$. This completes the proof of Theorem 8.1.

References

[1] D.Barbash and D.Vogan, The local structure of characters, J.Funct.Anal. **37** (1980), 27-55.

[2] I.N.Bernstein, Modules over a ring of differential operators: study of fundamental solutions of equations with constant coefficients, Funct.Anal.Appl. **5** (1971), 89-101.

[3] W.Borho and J.-L.Brylinski, Differential operators on homogeneous spaces I, Invent.math. **69** (1982), 437-476; III, ibid. **80** (1985), 1-68.

[4] M.Duflo, Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple, Ann.of Math. **105** (1977), 107-120.

[5] M.Duflo, Représentations de carré intégrable des groupes semi-simples réels, Exposé 508, Séminaire Bourbaki, 1977/78, pp.22-40.

[6] Harish-Chandra, Discrete series for semisimple Lie groups, I, II, Acta Math., **113** (1965), 241-318; ibid. **116** (1966), 1-111.

[7] G.Helgason, "Groups and Geometric Analysis," Academic Press, 1984.

[8] T.Hirai, Structure of unipotent orbits and Fourier transform of unipotent orbital integrals for semisimple Lie groups, *in* "Lectures on Harmonic Analysis on Lie Groups and Related Topics (T.Hirai and G.Schiffman eds.)", Lectures in Math., Kyoto Univ., **14**, 1982, pp.75-138.

[9] R.Hotta and R.Parthasarathy, Multiplicity formulae for discrete series, Invent.math. **26** (1974), 133-178.

[10] A.Joseph, Gelfand-Kirillov dimension for the annihilators of simple quotients of Verma modules, J.London Math.Soc. **18** (1978), 50-60.

[11] B.Kostant and S.Rallis, Orbits and representations associated with symmetric spaces, Amer.Math.J. **93** (1971), 753-809.

[12] H.Kraft and C.Procesi, Closures of conjugacy classes of matrices are normal, Invent.math. **53** (1979), 227-247.

[13] W.Schmid, Construction and classification of irreducible Harish-Chandra modules, *in* "Harmonic Analysis on Reductive Groups (W.Barker and P.Sally eds.)," Birkhäuser, 1991, pp.235-276.

[14] V.S.Varadarajan, Infinitesimal theory of representations of semisimple Lie groups, *in* "Harmonic Analysis and Representations of Semisimple Lie Groups (J.A.Wolf, M.Cahen and M.De Wilde eds.)," Reidel, 1980, pp.131-255.

[15] D.A.Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules, Invent.math. **48** (1978), 75-98.

[16] D.A.Vogan, Associated varieties and unipotent representations, *in* "Harmonic Analysis on Reductive Groups (W.Barker and P.Sally eds.)," Birkhäuser, 1991, pp.315-388.

[17] G.Warner, "Harmonic Analysis on Semi-simple Lie Groups I," Springer-Verlag, 1972.

[18] A.Yamaguchi, On higher-order terms in asymptotic expansions for irreducible characters of semisimple Lie groups, Master thesis, Kyoto University, 1986.

[19] H.Yamashita, Finite multiplicity theorems for induced representations of semisimple Lie groups I, J.Math.Kyoto Univ. **28** (1988), 173-211; II, ibid. 383-444.

[20] H.Yamashita, Embeddings of discrete series into induced representations of semisimple Lie groups, I, Japan.J.Math.(N.S.) **16** (1990), 31-95; II, J.Math.Kyoto Univ. **31** (1991), 543-571.

[21] H.Yamashita, Criteria for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras and applications to Harish-Chandra modules: study in relation with the associated varieties, to appear in J.Funct.Anal. (The principal results have been announced in Proc.Japan Acad. **68** (1992), 316-321.)

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