New applications of the geometric method of calculating syzygies.

Talk at the 6-th conference on nilpotent orbits and representation theory

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In this talk I describe two recent applications of the method.

We work over an algebraically closed field K of characteristic zero.

1. The geometric technique.

Let us consider the projective variety V of dimension m. For the purposes of this talk, $V = G/P$ for some reductive group G and parabolic subgroup P.

Let $X = A^N_{\mathbf{K}}$ be the affine space which is a representation of G. The space $X \times V$ can be viewed as a total space of trivial vector bundle $\mathcal{E}$ of dimension n over V. Let us consider the subvariety Z in $X \times V$ which is the total space of a subbundle $\mathcal{S}$ in $\mathcal{E}$. We denote by q the projection $q : X \times V \longrightarrow X$ and by q' the restriction of q to Z. Let $Y = q(Z)$. We get the basic diagram

$$\begin{array}{ccc} Z & \subset & X \times V \\ \downarrow q' & & \downarrow q \\ Y & \subset & X \end{array}$$

The projection from $X \times V$ onto V is denoted by p and the quotient bundle $\mathcal{E}/\mathcal{S}$ by $\mathcal{T}$. Thus we have the exact sequence of vector bundles on V

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0$$

The dimensions of $\mathcal{S}$ and $\mathcal{T}$ will be denoted by s, t respectively. The coordinate ring of X will be denoted by A. It is a polynomial ring in N variables over $\mathbf{K}$. We will identify the sheaves on X with A-modules.

The locally free resolution of the sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_{X\times V}$-module is given by the Koszul complex

$$\mathcal{K}_\bullet(\xi) : 0 \to \bigwedge^t (p^*\xi) \to \ldots \to \bigwedge^2 (p^*\xi) \to p^*(\xi) \to \mathcal{O}_{X\times V}$$

where $\xi = \mathcal{T}^*$. The differentials in this complex are homogeneous of degree 1 in the coordinate functions on X. The direct image $p_*(\mathcal{O}_Z)$ can be identified with the the sheaf of algebras $Sym(\eta)$ where $\eta = \mathcal{S}^*$.

The idea of the geometric technique is to use the Koszul complex $\mathcal{K}(\xi)_\bullet$ to construct for each vector bundle $\mathcal{V}$ on V the free complex $\mathbf{F}_\bullet(\mathcal{V})$ of A-modules with the homology supported in Y. In many cases the complex $\mathbf{F}(\mathcal{O}_V)_\bullet$ gives the free resolution of the defining ideal of Y.

For every vector bundle $\mathcal{V}$ on V we introduce the complex

$$\mathcal{K}(\xi, \mathcal{V})_\bullet := \mathcal{K}(\xi)_\bullet \otimes_{\mathcal{O}_{X\times V}} p^*\mathcal{V}$$

This complex is a locally free resolution of the $\mathcal{O}_{X\times V}$-module $M(\mathcal{V}) := \mathcal{O}_Z \otimes p^*\mathcal{V}$.

Now we are ready to state the basic theorem (Theorem (5.1.2) in [W]).

Theorem 1. *For a vector bundle $\mathcal{V}$ on V we define a free graded A-modules*

$$\mathbf{F}(\mathcal{V})_i = \bigoplus_{j\geq 0} H^j(V, \bigwedge^{i+j} \xi \otimes \mathcal{V}) \otimes_k A(-i-j)$$

a) *There exist minimal differentials*

$$d_i(\mathcal{V}) : \mathbf{F}(\mathcal{V})_i \to \mathbf{F}(\mathcal{V})_{i-1}$$

of degree 0 such that $\mathbf{F}(\mathcal{V})_\bullet$ is a complex of graded free A-modules with

$$H_{-i}(\mathbf{F}(\mathcal{V})_\bullet) = \mathcal{R}^i q_* M(\mathcal{V})$$

In particular the complex $\mathbf{F}(\mathcal{V})_\bullet$ is exact in positive degrees.

b) *The sheaf $\mathcal{R}^i q_* M(\mathcal{V})$ is equal to $H^i(Z, M(\mathcal{V}))$ and it can be also identified with the graded A-module $H^i(V, Sym(\eta) \otimes \mathcal{V})$.*

c) *If $\phi : M(\mathcal{V}) \to M(\mathcal{V}')(n)$ is a morphism of graded sheaves then there exists a morphism of complexes*

$$f_\bullet(\phi) : \mathbf{F}(\mathcal{V})_\bullet \to \mathbf{F}(\mathcal{V}')_\bullet(n)$$

Its induced map $H_{-i}(f_\bullet(\phi))$ can be identified with the induced map

$$H^i(Z, M(\mathcal{V})) \to H^i(Z, M(\mathcal{V}'))(n).$$

If $\mathcal{V}$ is a one dimensional trivial bundle on V then the complex $\mathbf{F}(\mathcal{V})_\bullet$ is denoted simply by $\mathbf{F}_\bullet$.

The next theorem gives the criterion for the complex $\mathbf{F}_\bullet$ to be the free resolution of the coordinate ring of Y.

Theorem 2. *Let us assume that the map $q' : Z \longrightarrow Y$ is a birational isomorphism. Then the following properties hold.*

a) *The module $q'_* \mathcal{O}_Z$ is the normalization of $\mathbf{K}[Y]$.*

b) *If $\mathcal{R}^i q'_* \mathcal{O}_Z = 0$ for $i > 0$, then $\mathbf{F}_\bullet$ is a finite free resolution of the normalization of $\mathbf{K}[Y]$ treated as an A-module.*

c) *If $\mathcal{R}^i q'_* \mathcal{O}_Z = 0$ for $i > 0$ and $\mathbf{F}_0 = H^0(V, \bigwedge^0 \xi) \otimes A = A$ then Y is normal and it has rational singularities.*

This is Theorem (5.1.3) in [W].

This technique was successfully applied to determinantal varieties related to generic, symmetric and skew-symmetric matrices. Also interesting results regarding defining ideals of nilpotent orbits were obtained. The determinantal expressions for resultants and discriminants were also obtained usiing this method. All these developments are described in [W].

One should point out that the terms of the complex $\mathbf{F}_\bullet$ can be calculated fully only when the bundle ξ has convenient form (for example is a tensor product of tautological bundles on some Grassmannians). We will refer to this saying that ξ is in a simple form.

Notation. *In the examples below, for the integers a_i with $a_1 \geq \ldots \geq a_n \geq 0$, we denote $S_{(a_1,\ldots,a_n)}E$ the highest weight representation of the group $GL(E)$ corresponding to the weight $(a_1,\ldots,a_n)$, i.e. the Schur functor.*

2. Representations with finitely many orbits.

The natural scope for application of this method is afforded by the orbit closures in the irreducible representations of simple groups with finitely many orbits. These were classified by Kac [K] and, with some minor corrections added, by Leahy [Le].

The first type of such representations (Table II in [K]) is the family of representations parametrized by a Dynkin diagram with the distinguished (black) node. Distinguishing such a node determines a grading on the corresponding root system, in which all simple roots except the one corresponding to the black node, have degree 0 and the root corresponding to the black node has degree 1. This leads to a grading

$$\underline{g} = \oplus_{i\in\mathbf{Z}}\underline{g}_i$$

of the Lie algebra $\underline{g}$ corresponding to our Dynkin diagram. The representation we are interested in is the representation of G_0 (the group corresponding to the Lie algebra $\underline{g}_0$ on $\underline{g}_1$. The orbits of such actions were classified by Vinberg in [V]. They are the irreducible components of intersections of nilpotent orbits in $\underline{g}$ with $\underline{g}_1$.

For Dynkin diagrams of classical types, most of the representations one gets are of the type $Hom(E,F)$ where E is a vector space and F is a symplectic (resp. orthogonal vector space), with the action of the group $GL(E)\times SP(F)$ (resp. $GL(E)\times SO(F)$).

Last year Steve Lovett obtained in his thesis [L] some very interesting results on such orbit closures. Let me describe his results.

Consider the representation $Hom(E,F)$. We denote $dim\ E = e$, $dim\ F = f$. The orbits in such representations are described by two rank conditions: the rank r_1 of the map $\phi\in Hom(E,F)$ and the rank r_2 of the form on F restricted to $m(\phi)$. We denote the corresponding orbit O_{r_1,r_2} and the orbit closure $Z_{r_1,r_2} := \bar{O}_{r_1,r_2}$.

Lovett found desingularizations of the orbit closures Z_{r_1,r_2} for which geometric technique is applicable. Unfortunately, the bundles ξ are not very simple, they are extensions of two tensor products. Lovett first classifies the orbit closures for which one can find a desingularization with ξ in a simple form.

Definition. *An orbit closure Z_{r_1,r_2} is special if one of the following conditions holds.*

a) $r_2 = 0$,

b) $r_1 = r_2$,

c) $r_1 = e$,

d) $2r_1 - r_2 = f$.

For special orbits the terms of the complexes $\mathbf{F}_\bullet$ are calculated explicitly thus providing minimal free resolutions of the coordinate rings.

In particular we get

Theorem 3 (Lovett). *Let E be a vector space, F- a symplectic or orthogonal vector space. Let Z_{r_1,r_2} be a special orbit closure. Then Z_{r_1,r_2} is normal, with rational singularities (hence Cohen-Macaulay), except in the case when F is orthogonal and case d) above occurs (with $r_2 \neq 0$ if f is even). In these bad cases the normalization of the coordinate ring has rational singularities, but the coordinate ring itself is not Cohen-Macaulay.*

One should add that the defining ideals for normal special orbit closures are given by *natural rank conditions*, i.e. by $r_1 + 1$ minors of ϕ and by the corresponding pfaffians (resp. minors) of the skew-symmetric (resp. symmetric) matrix given by $SP(F)$ (resp. $SO(F)$)-invariants in A of degree 2.

The next question are the properties of nonspecial orbits.

Theorem 4 (Lovett). *Let E be a vector space, F- a symplectic or orthogonal vector space. Let Z_{r_1,r_2} be a nonspecial orbit closure. Then the coordinate ring of Z_{r_1,r_2} is Cohen-Macaulay. Moreover, the defining ideal of Z_{r_1,r_2} is given by natural rank conditions.*

Sketch of proof Consider the Grassmannian $Grass(e - r_1, E)$ and the desingularization of the determinantal variety

$$Z = \{(\phi, R) \in X \times Grass(e - r_1, E) \mid \phi|_R = 0\}.$$

Now, take the relative version of the resulution of the orbit closure O_{r_1,r_2} in $Hom(E'F)$ with $dim\ E' = r_1$, and changing E' to the tautological factor bundle $\mathcal{Q}$. Notice that this orbit was special (case c)), so it has rational singularities and the resolution could be effectively described. Taking the direct image over $Grass(e - r_1, E)$ of all the terms, we get a complex of sheaves of $\mathcal{B}$-modules where $\mathcal{B} = Sym(\mathcal{Q} \otimes F)$ is a sheaf of algebras. Moreover, the terms of the resolution are the direct sumes of sheaves of type $S_\lambda \mathcal{Q} \otimes \mathcal{B}$. Let us denote the i-th term of this complex by $\mathcal{F}_i$ $(0 \leq i \leq d)$. Here d is the codimension of Z_{r_1,r_2} in $Hom(E', F)$.

The key observation is as follows

Fact.

a) $H^j(Grass(e - r_1, E), \mathcal{F}_i) = 0$ *for* $j > 0$, $0 \leq i \leq d$,

b) $H^0(Grass(e - r_1, E), \mathcal{F}_i)$ *is a maximal Cohen-Macaulay module supported in determinantal variety* Z_{r_1,r_1}.

The second fact is proved by geometric method for twisted modules supported in the determinantal variety.

This result implies that we can construct the resolution of the cokerel of the exact sequence

$$0 \to H^0(Grass(e-r_1, E), \mathcal{F}_d) \to \ldots \to$$

$$\to H^0(Grass(e-r_1, E), \mathcal{F}_1) \to H^0(Grass(e-r_1, E), \mathcal{F}_0)$$

by iterated cone construction from the terms of resolutions of $H^0(Grass(e-r_1, E), \mathcal{F}_i)$. This gives a nonminimal resolution of the coordinate ring of Z_{r_1,r_2} of the length equal to its codimension.

The claim about the equations comes from analysing the low degree terms in the complex. After we establish that the generators correspond to natural equations giving rank conditions, we notice that since the ideal is perfect, in order to prove it is reduced, it is enough to show it is generically reduced. This can be then done explicitly by localization.

3. The secant varieties.

I am collaborating with Joe Landsberg and Laurent Manivel on secant varieties for multiple Segre embeddings. These were studied in the past only regarding their dimensions. Other properties and defining equations were not investigated. We are dealing here with the tensor product $F_1 \otimes \ldots \otimes F_n$ and the subvariety X of totally decomposable tensors. The variety $Sec(X)$ is the closure of the set of tensors that can be written as a sum of two totally decomposable tensors.

Landsberg and Manivel observed in [LM] that the geometric method can be applied. Using the fact that the secant variety $Sec(X)$ in $F_1 \otimes F_2 \otimes F_3$, $dim\ F_i = 2$ for $1 \le i \le 3$ is the whole space $F_1 \otimes F_2 \otimes F_3$ they proved

Theorem 5 (Landsberg-Manivel). *Let us consider the secant variety $Sec(X)$ in $F_1 \otimes F_2 \otimes F_3$. It is normal, with rational singularities. The defining ideal of $Sec(X)$ is generated in degree 3. The generators are 3×3 minors of linear maps $\phi_{i,j} : F_i \otimes F_j \otimes A(-1) \to F_k^* \otimes A$ (here $\{i,j,k\} = \{1,2,3\}$) induced by $\phi \in F_1 \otimes F_2 \otimes F_3$.*

This case might seem special because of the special property of the secant variety for 3 copies of K^2 we used.

We analyzed the case of four copies.

Theorem 6 (Landsberg-Manivel-JW). *Let us consider the secant variety $Sec(X)$ in $F_1 \otimes F_2 \otimes F_3 \otimes F_4$. It's coordinate ring is Cohen-Macaulay. The defining ideal of $Sec(X)$ is generated in degree 3. The generators are 3×3 minors of linear maps $\phi_{i,j} : F_i \otimes F_j \otimes A(-1) \to F_k^* \otimes F_l^* \otimes A$ (here $\{i,j,k,l\} = \{1,2,3,4\}$) induced by $\phi \in F_1 \otimes F_2 \otimes F_3$.*

Sketch of proof The proof is very similar to the proof of theorem 4.

Consider the product of Grassmannians $\prod_{i=1}^{4} Grass(f_i - 2, F_i)$ and the desingularization of the corresponding variety Z, which has

$$p_*(\mathcal{O}_Z = Sym(\otimes_{i=1}^{4} \mathcal{Q}_i)$$

where

$$0 \to \mathcal{R}_i \to F_i \otimes Grass(f_i - 2, F_i) \to \mathcal{Q}_i \to 0$$

is a tautological sequence on $Grass(f_i - 2, F_i)$, with $f_i = dim\ F_i$. Now, take the relative version of the resulution of the variety $Sec(X)$ in product of 4 copies of two dimensional spaces. This is a variety of codimension 6 and the resolution of its coordinate ring can be calculated using Macaulay 2.

We get a complex of sheaves of $\mathcal{B}$-modules with the terms being the direct sumes of sheaves of type $\otimes_{i=1}^{4} S_{a_i,b_i)} \mathcal{Q}_i \otimes \mathcal{B}$. Let us denote the i-th term of this complex by $\mathcal{F}_i$ $(0 \leq i \leq 6)$.

The key observation is again

Fact.

a) $H^j(Grass(e - r_1, E), \mathcal{F}_i) = 0$ *for* $j > 0$, $0 \leq i \leq 6$,

b) $H^0(Grass(e - r_1, E), \mathcal{F}_i)$ *is a maximal Cohen-Macaulay module supported in the variety* Z.

Now we can repeat the reasoning used to prove theorem 4.

We have similar results for the second secant variety in the Segre embedding of a triple product of projective spaces.

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