Two-sided cells and equivariant K-theory

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1. Hecke algebra

G : connected, simply-connected
$\bigcup$ simple algebraic group / $\mathbb{C}$

B : Borel subgroup
$\cup$
T : maximal torus

$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{b} = \mathrm{Lie}\, B$, $\mathfrak{t} = \mathrm{Lie}\, T$

Δ : root system ($\subset \mathfrak{t}^*$)
$\cup$
$\Delta^+ = \{\text{positive root}\} = \{\text{weight of } \mathfrak{g}/\mathfrak{b}\}$
$\cup$
$\Pi = \{\text{simple root}\}$

Q : root lattice
$\cap$
P : weight lattice
$\cup$
$P^+ = \{\text{dominant weights}\}$

W : Weyl group

$W_a := W \ltimes Q$ (affine Weyl group)

$\cap$

$\tilde{W}_a := W \ltimes P$ ((extended) affine Weyl group)

$$\tilde{W}_a = \{ w t_\lambda \mid w \in W, \lambda \in P \}$$

$$(w t_\lambda)(y t_\mu) = w y \, t_{y^{-1}\lambda + \mu}$$

- W_a is a Coxeter group with canonical generator system

$$S_a = \{ s_\alpha \mid \alpha \in \Pi \} \sqcup \{ t_\theta s_\theta \}$$

$\theta \in \Delta^+$ s.t. $\theta^\vee$ is highest coroot

- $\tilde{W}_a = \Omega \ltimes W_a$ with

$$\Omega = \{ x \in \tilde{W}_a \mid x S_a = S_a x \}$$

- The Bruhat order $\geqq$, the length function $l : W_a \longrightarrow \mathbb{Z}_{\geq 0}$ on W_a are extended to $\tilde{W}_a$ by
  - $\omega x \geqq \omega' x' \Leftrightarrow \omega = \omega'$, $x \geqq x'$
    $(\omega, \omega' \in \Omega,\ x, x' \in W_a)$
  - $l(\omega x) = l(x) \qquad (\omega \in \Omega,\ x \in W_a)$

<u>def</u> $H(\tilde{W}_a)$: associative algebra $/ \overset{A}{\overset{\|}{\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]}}$

- $\{ T_x \mid x \in \tilde{W}_a \}$ free A-basis

<u>relations</u>

$$T_x T_{x'} = T_{xx'} \quad (\text{if } l(xx') = l(x) + l(x'))$$

$$(T_s + 1)(T_s - q) = 0 \qquad (s \in S_a)$$

(remark: $T_e = 1$)

Another presentation

$$H(\widetilde{W}_a) = \bigoplus_{\substack{w \in W \\ \lambda \in P}} A\, T_w \theta_\lambda$$

$$\begin{cases} \theta_0 = 1 \\ \theta_\lambda \theta_\mu = \theta_{\lambda+\mu} \quad (\lambda, \mu \in P) \\ T_{s_\alpha} \theta_\lambda = \theta_{s\lambda} T_{s_\alpha} + (q-1) \dfrac{\theta_\alpha (\theta_\lambda - \theta_{s\lambda})}{\theta_\alpha - 1} \end{cases}$$

$$(\alpha \in \Pi,\ \lambda \in P)$$

$$\theta_\lambda = q^{(-\ell(t_{\lambda_1}) + \ell(t_{\lambda_2}))/2}\, T_{t_{\lambda_1}} T_{t_{\lambda_2}}^{-1}$$

with

$$\lambda = \lambda_1 - \lambda_2, \quad \lambda_1, \lambda_2 \in P^+$$

2. Kazhdan-Lusztig basis and 2-sided cells

<u>Proposition</u> (Kazhdan-Lusztig 1979)

$\forall x \in \widetilde{W}_a \quad \exists! \; C_x \in H(\widetilde{W}_a)$ s.t.

$$C_x = \sum_{y \leq x} P_{y,x}(q) \, T_y \quad \text{with}$$

(a) $P_{x,x}(q) = 1$

(b) for $y < x$ $\quad P_{y,x}(q) \in \mathbb{Z}[q]$ with

$$\text{degree}\,(P_{y,x}(q)) \leq \frac{\ell(x) - \ell(y) - 1}{2}$$

(c) $$C_x = q^{\ell(x)} \sum_{y \leq x} P_{y,x}(q^{-1}) \, T_{y^{-1}}^{-1}$$ ┘

- $\{ C_x \mid x \in \widetilde{W}_a \}$ is a free A-basis of $H(\widetilde{W}_a)$, called the Kazhdan-Lusztig basis.

<u>def</u> For $x \in \widetilde{W}_a$

I_x : the smallest two sided ideal of $H(\widetilde{W}_a)$ s.t.

- $I_x \ni C_x$
- I_x is spanned over A by a subset of $\{C_y \mid y \in \widetilde{W}_a\}$

<u>def</u>

$$y \underset{LR}{\sim} x \iff I_y = I_x$$

(equivalence relation on $\widetilde{W}_a$)

Equivalence classes w.r.t. $\underset{LR}{\sim}$ are called <u>two-sided cells</u>.

For a two-sided cell $\mathbf{c}$ set

$$I_{\mathbf{c}} = I_x \quad (\text{for } x \in \mathbf{c})$$

Theorem (Lusztig 1989)

$$\{\text{Two-sided cell}\} \simeq \{\text{nilpotent orbit in } \mathfrak{g}\}.$$

Notation

For a nilpotent orbit $\mathcal{O}$ denote by $\mathcal{C}_{\mathcal{O}}$ the corresponding two sided cell.

Conjecture $I_{\mathcal{C}_{\mathcal{O}}} \supset I_{\mathcal{C}_{\mathcal{O}'}} \Leftrightarrow \overline{\mathcal{O}} \supset \mathcal{O}'$

3. Equivariant K-theory

$$\mathcal{B} = G/B = \{\text{Borel subalgebra of } \mathfrak{g}\}$$

$$\begin{array}{ccc} \mathcal{B} & & \{\ldots\} \\ \Psi & & \Psi \\ x & \longleftrightarrow & \mathfrak{b}_x \\ & & \cup \\ & & \mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x] \end{array}$$

$$\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} Y_w \qquad Y_w = G(eB, wB)$$

$$T^*\mathcal{B} \cong \{(a, x) \in \mathfrak{g} \times \mathcal{B} \mid a \in \mathfrak{n}_x\}$$

$$Z = \coprod_{w \in W} T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \subset T^*(\mathcal{B} \times \mathcal{B}) = T^*\mathcal{B} \times T^*\mathcal{B}$$

$$Z = \{((a, x), (a', x')) \in T^*\mathcal{B} \times T^*\mathcal{B} \mid a + a' = 0\}$$
$$\cong \{(a, x, x') \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid a \in \mathfrak{n}_x \cap \mathfrak{n}_{x'}\}$$

$G \times \mathbb{C}^\times$ acts on $T^*\mathcal{B}$ by

$$(g, z) : (a, x) \longmapsto (z^{-2} \operatorname{Ad}(g) a, gx)$$

It induces actions of $G \times \mathbb{C}^\times$ on $T^*\mathcal{B} \times T^*\mathcal{B}$, Z

$K^{G\times\mathbb{C}^\times}(Z)$

:= the Grothendieck group of

$\left\{ \begin{array}{l} G\times\mathbb{C}^\times\text{-equivariant coherent } \mathcal{O}_{T^*\mathcal{B}\times T^*\mathcal{B}}\text{-module } M \\ \text{with } \operatorname{supp}(M)\subset Z \end{array} \right\}$

- $K^{G\times\mathbb{C}^\times}(Z)$ is an A-module

  ($R^{\mathbb{C}^\times}$ = (representation ring of $\mathbb{C}^\times$) $\simeq \mathbb{Z}[q^{\pm\frac{1}{2}}] = A$)

- $K^{G\times\mathbb{C}^\times}(Z)$ is an A-algebra w.r.t. the convolution product:

$$m * n := p_{13*}(p_{12}^* m \otimes p_{23}^* n)$$

$$p_{ij} : T^*\mathcal{B}\times T^*\mathcal{B}\times T^*\mathcal{B} \longrightarrow T^*\mathcal{B}\times T^*\mathcal{B}$$

(projection)

<u>Theorem</u> (Kazhdan–Lusztig, Ginzburg 1987)

There exists an isomorphism

$$\Phi : H(\tilde{W}_a) \xrightarrow{\sim} K^{G\times\mathbb{C}^\times}(Z)$$

of A-algebras

s.t.

$$\Phi(\theta_\lambda) = [r_{1*}\pi_1^*\mathcal{O}_{Y_e}(-\lambda)] \quad (\lambda \in P)$$

$$\Phi(T_{s_\alpha}+1) = -[r_{s*}\pi_s^*(\Omega_{\overline{Y_s}} \otimes \lambda_s^*(\mathcal{O}_{\mathcal{B}} \boxtimes \Omega_{\mathcal{B}}^{\otimes -1}))]$$

$$\begin{tikzcd} \overline{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})} \arrow[r, hook, "r_w"] \arrow[d, "\pi_w"'] & Z \arrow[d] \\ \overline{Y_w} \arrow[r, hook, "i_w"'] & \mathcal{B}\times\mathcal{B} \end{tikzcd}$$

$Y_e = \overline{Y_e} \simeq \mathcal{B}$

Problem

Describe $\Phi(C_x)$.

Conjecture (#)

For any nilpotent orbit $\mathcal{O}$

$$\begin{tikzcd}
H(\tilde{W}_a) \arrow[d, "\Phi"', "\wr"] & \supset & I_{e_{\mathcal{O}}} \arrow[d, "\wr"'] \\
K^{G\times\mathbb{C}^*}(Z) & \supset & K^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{O}}})
\end{tikzcd}$$

$$Z_{\bar{\mathcal{O}}} = \{(a, x, y) \in Z \mid a \in \bar{\mathcal{O}}\}$$

4. A description of $\Phi(C_w)$ for $w \in W$

For a smooth algebraic variety X over $\mathbb{C}$ we have a category

$$MHM(X) = \{\text{mixed Hodge module on } X\}$$

$$\subset \left\{ (M, F, K, W) \;\middle|\; \begin{array}{l} M : \text{regular holonomic } D_X\text{-module} \\ F : \text{good filtration of } M \\ K \in \mathrm{Perv}(\mathbb{Q}_X) \text{ s.t. } \mathbb{C} \otimes_{\mathbb{Q}} K = DR(M) \\ W : \text{filtration of } (M, F, K) \end{array} \right\}$$

For $\mathcal{M} = (M, F, K, W) \in MHM(X)$ we have a coherent $\mathcal{O}_{T^*X}$-module

$$gr\, \mathcal{M} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}(gr D_X)} \pi^{-1}(gr^F M)$$

$$\pi : T^*X \longrightarrow X.$$

$$MHM^{G}(\mathcal{B}\times\mathcal{B}) = \left\{ \begin{array}{l} \text{mixed Hodge module on } \mathcal{B}\times\mathcal{B} \\ \text{with } G\text{-action} \end{array} \right\}$$

$$K(MHM^{G}(\mathcal{B}\times\mathcal{B})) \xrightarrow{gr} K^{G\times\mathbb{C}^{*}}(Z)$$

- $K(MHM^{G}(\mathcal{B}\times\mathcal{B}))$ is equipped with an A-algebra structure via the convolution product

$$[m]\cdot[n] = [g_{13*}(g_{12}^{*}m \otimes g_{23}^{*}n)]$$

$g_{ij} : \mathcal{B}\times\mathcal{B}\times\mathcal{B} \to \mathcal{B}\times\mathcal{B}$ projection

<u>Theorem (T, 1987)</u>

$$\begin{array}{ccc} H(W) & \hookrightarrow & H(\tilde{W}_a) \\ \wr\! | & & \wr\! \downarrow \Phi \\ K(MHM^{G}(\mathcal{B}\times\mathcal{B})) & \xrightarrow{gr} & K^{G\times\mathbb{C}^{*}}(Z) \end{array}$$

$$\Phi(C_w) = gr([{}^{\pi}\mathcal{O}^{H}_{Y_w}]) \quad (w\in W)$$

<u>Corollary</u>

$$w\in W \Longrightarrow \Phi(C_w) \in K^{G\times\mathbb{C}^{*}}(Z_{\bar{o}})$$

$$(w \in \mathcal{C}_o \quad)$$

5. The case $G = SL_n(\mathbb{C})$

Theorem (T–Xi = 席)

Conjecture (#) holds for $G = SL_n(\mathbb{C})$, i.e.

$$\begin{tikzcd} H(\tilde{W}_a) \arrow[d, "\Phi", "\wr"'] & \supset & I_{c_0} \arrow[d, "\wr"] \\ K^{G\times\mathbb{C}^*}(Z) & \supset & K^{G\times\mathbb{C}^*}(Z_{\bar{O}}) \end{tikzcd}$$

Outline of the proof

Fact 1 (Shi = 時)

$$I_{c_0} \supset I_{c_{0'}} \Longleftrightarrow \bar{O} \supset O'$$

So we can use induction on $\dim O$

Fact 2 $W \cap c_0 \neq \phi$ for any O

Fact 3 $I_{c_0} / \sum_{O' \subset \bar{O}} I_{c_{0'}}$ is generated as an $H(\hat{W}_a)$-bimodule by a single element C_w ($w \in W \cap c_0$).

$$\Phi(C_w) \in K^{G\times\mathbb{C}^\times}(Z_{\bar{O}}).$$

Hence by Fact 3 we have

$$\Phi(I_{e_0}) \subset K^{G\times\mathbb{C}^\times}(Z_{\bar{O}}).$$

"$\supset$" can be proved using representation theory of $\overline{\mathbb{C}(q)} \otimes_A H(\tilde{W}_a)$ due to Kazhdan–Lusztig and Ginzburg. //