THE CAPELLI HARISH-CHANDRA HOMOMORPHISM IS COMPATIBLE WITH THE CAUCHY HARISH-CHANDRA INTEGRAL.

Tomasz Przebinda
Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA

1. Dual pairs, correspondence of the semisimple orbits, and the Capelli Harish-Chandra homomorphism.

Here we recall some notation and results from [P3]. Let $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let V_0, V_1 be two finite dimensional left vector spaces over $\mathbb{D}$. Set

$$V = V_0 \oplus V_1 \tag{1.1}$$

and define an element $s \in End(V)$ by

$$s(v_0 + v_1) = v_0 - v_1 \qquad (v_o \in V_0,\ v_1 \in V_1). \tag{1.2}$$

Set

$$\begin{aligned} End(V)_0 &= \{x \in End(V);\ sx = xs\}, \\ End(V)_1 &= \{x \in End(V);\ sx = -xs\}, \\ GL(V)_0 &= GL(V) \cap End(V)_0. \end{aligned} \tag{1.3}$$

The real vector space $End(V)_0$ is a Lie algebra, with the usual commutator $[x, y] = xy - yx$. The action of $GL(V)_0$, by conjugation, on $End(V)$

$$Conj(g)x = gxg^{-1} \qquad (g \in GL(V)_0,\ x \in End(V))$$

preserves both $End(V)_0$ and $End(V)_1$. Furthermore, the anticommutator

$$End(V)_1 \times End(V)_1 \ni (x, y) \to \{x, y\} = xy + yx \in End(V)_0 \tag{1.4}$$

is $\mathbb{R}$-bi-linear and $GL(V)_0$-equivariant. Set

$$\langle x, y\rangle = tr_{\mathbb{D}/\mathbb{R}}\{sx, y\} \qquad (x, y \in End(V)). \tag{1.5}$$

It is easy to see that the form (1.5) is preserved under the action of $GL(V)_0$.

Lemma 1.6. *If $V_0 \neq 0$ and $V_1 \neq 0$ then the restriction of the bilinear form $\langle\ ,\ \rangle$ to $End(V)_1$ is symplectic and non-degenerate. Moreover, the group homomorphism*

$$Conj : GL(V)_0 \to Sp(End(V)_1, \langle\ ,\ \rangle)$$

maps the groups

$$GL(V)_0|_{V_0} = \{g \in GL(V)_0;\ g|_{V_1} = 1\},$$
$$GL(V)_0|_{V_1} = \{g \in GL(V)_0;\ g|_{V_0} = 1\}$$

injectively onto an irreducible dual pair of type II in the symplectic group $Sp(End(V)_1, \langle\ ,\ \rangle)$.

Let ι be a possibly trivial involution on $\mathbb{D}$. Let ϕ_0 be a non-degenerate ι-hermitian form on V_0, and let ϕ_1 be a non-degenerate ι-skew-hermitian form on V_1. Set $\phi = \phi_0 \oplus \phi_1$. Define

$$\begin{aligned} &\mathfrak{s}(V,\phi)_0 = \{x \in End(V)_0;\ \phi(xu, v) = \phi(u, -xv),\ u, v \in V\},\\ (1.7)\qquad &\mathfrak{s}(V,\phi)_1 = \{x \in End(V)_1;\ \phi(xu, v) = \phi(u, sxv),\ u, v \in V\},\\ &S(V,\phi)_0 = \{g \in GL(V)_0;\ \phi(gu, gv) = \phi(u, v),\ u, v \in V\}. \end{aligned}$$

Clearly, $S(V,\phi)_0$ is a Lie subgroup of $GL(V)_0$, with the Lie algebra $\mathfrak{s}(V,\phi)_0$. Moreover, it is easy to check that the anticommutator (1.4) maps $\mathfrak{s}(V,\phi)_1 \times \mathfrak{s}(V,\phi)_1$ into $\mathfrak{s}(V,\phi)_0$. Furthermore, the adjoint action of $S(V,\phi)_0$ preserves $\mathfrak{s}(V,\phi)_0$, $\mathfrak{s}(V,\phi)_1$, and the form (1.5).

Lemma 1.8. *If $V_0 \neq 0$ and $V_1 \neq 0$ then the restriction of the bilinear form $\langle\ ,\ \rangle$ to $\mathfrak{s}(V,\phi)_1$ is symplectic and non-degenerate. Moreover,*

$$Conj : S(V,\phi)_0 \to Sp(\mathfrak{s}(V,\phi)_1, \langle\ ,\ \rangle)$$

maps the groups

$$S(V,\phi)_0|_{V_0} = \{g \in S(V,\phi)_0;\ g|_{V_1} = 1\},$$
$$S(V,\phi)_0|_{V_1} = \{g \in S(V,\phi)_0;\ g|_{V_0} = 1\}$$

injectively onto an irreducible dual pair of type I in the symplectic group $Sp(\mathfrak{s}(V,\phi)_1, \langle\ ,\ \rangle)$.

Definition 1.9. *An irreducible ordinary classical Lie supergroup is a pair $(S, \mathfrak{s})$ with $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$, where either*

(I) $\qquad S = S(V,\phi)_0,\ \mathfrak{s}_0 = \mathfrak{s}(V,\phi)_0,\ \mathfrak{s}_1 = \mathfrak{s}(V,\phi)_1,$ *as in (1.8),* *or*

(II) $$S = GL(V)_0,\ \mathfrak{s}_0 = End(V)_0,\ \mathfrak{s}_1 = End(V)_1,\ \textit{as in (1.3).}$$

In order to simplify the notation we shall write gx instead of $Conj(g)x$, for $g \in S$ and $x \in \mathfrak{s}_1$, and similarly for x^2. For $x \in \mathfrak{s}_1$ define the anticommutant of x in $\mathfrak{s}_1$ by

$$^x\mathfrak{s}_1 = \{z \in \mathfrak{s}_1;\ \{x, z\} = 0\}.$$

Definition 1.10. *The element $x \in \mathfrak{s}_1$ is called semi-simple if and only if x is semi-simple as an endomorphism of V. An element $x \in \mathfrak{s}_1$ is regular if and only if the S-orbit through x is of maximal possible dimension. A Cartan subspace $\mathfrak{h}_1 \subseteq \mathfrak{s}_1$ is the double anticommutant $\mathfrak{h}_1 = {}^{(^x\mathfrak{s}_1)}\mathfrak{s}_1$ of a regular semisimple element of $x \in \mathfrak{s}_1$.*

Let $\mathfrak{h}_1 \subseteq \mathfrak{s}_1$ be a Cartan subspace and let $\mathfrak{h}_1^{reg} \subseteq \mathfrak{h}_1$ be the subset of regular elements. Denote by $\mathfrak{h}_1^2 = \{\mathfrak{h}_1, \mathfrak{h}_1\}$ the linear span of the elements $\{x, y\}$, where $x, y \in \mathfrak{h}_1$. (This is the same as the span of all the squares, $x^2 = \frac{1}{2}\{x, x\}$, $x \in \mathfrak{s}_1$, hence our notation $\mathfrak{h}_1^2$.)

The following proposition may be viewed as "a linearization" of the correspondence of the semisimple orbits for dual pairs. The injectivity of this correspondence was verified by Roger Howe, with the proof based on Witt's Cancellation Theorem, [J].

Proposition 1.11. *The relation*

(a) $$\{(x^2|_{V_0}, x^2|_{V_1});\ x \in \mathfrak{h}_1\} \subseteq \mathfrak{h}_1^2|_{V_0} \times \mathfrak{h}_1^2|_{V_1}$$

is an invertible function, which extends uniquely to a linear bijection

(b) $$\mathfrak{h}_1^2|_{V_0} \leftrightarrow \mathfrak{h}_1^2|_{V_1}.$$

Suppose the rank of $S|_{V_i}$ is less than or equal to the rank of $S|_{V_j}$, $\{i, j\} = \{0, 1\}$. Then $\mathfrak{h}_1^2|_{V_i}$ is a Cartan subalgebra of $\mathfrak{s}_0|_{V_i}$.

In order to compress the notation we shall write

$$G = S|_{V_j},\ \mathfrak{g} = \mathfrak{s}_0|_{V_j},\ G' = S|_{V_i},\ \mathfrak{g}' = \mathfrak{s}_0|_{V_i}, \mathfrak{h}' = \mathfrak{h}_1^2|_{V_i}.$$

We identify

$$\mathfrak{h}' = \mathfrak{h}_1^2|_{V_j}, \tag{1.12}$$

by (1.11.b).

Let $\mathfrak{z}$ be the centralizer of $\mathfrak{h}'$ in $\mathfrak{g}$. The $\mathfrak{h}' \subseteq \mathfrak{z}$. Let $\mathfrak{z}''$ be the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{z}$, so that $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}''$. In particular we have

$$\mathcal{U}(\mathfrak{z}) = \mathcal{U}(\mathfrak{h}') \otimes \mathcal{U}(\mathfrak{z}''). \tag{1.13}$$

Let $\mathfrak{h}'' \subseteq \mathfrak{z}''$ be a Cartan subalgebra. Then $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}''$ is a Cartan subalgebra of $\mathfrak{g}$. Also, let

$$Z = \text{the centralizer of } \mathfrak{h}' \text{ in } G,$$

$$Z'' = \text{the isometry group with the Lie algebra } \mathfrak{z}'',$$

$$W' = \text{the Weyl group for } (G'_{\mathbb{C}},\, \mathfrak{h}'), \text{ and}$$

$$W = \text{the Weyl group for } (G_{\mathbb{C}},\, \mathfrak{h}).$$

Denote by

$$\epsilon : \mathcal{U}(\mathfrak{z}'') \to \mathbb{C}$$

the augmentation homomorphism (The derivative of the trivial representation of Z''). Then, by (1.13),

$$1 \otimes \epsilon : \mathcal{U}(\mathfrak{z}) \to \mathcal{U}(\mathfrak{h}').$$

The **Capelli Harish-Chandra homomorphism** is defined by

$$\mathcal{C} : \mathcal{U}(\mathfrak{g})^G \underset{\gamma_{\mathfrak{g}/\mathfrak{h}}}{\to} \mathcal{U}(\mathfrak{h})^W \underset{\gamma_{\mathfrak{z}/\mathfrak{h}}^{-1}}{\to} \mathcal{U}(\mathfrak{z})^Z \underset{1\otimes\epsilon}{\to} \mathcal{U}(\mathfrak{h}')^{W'} \underset{\gamma_{\mathfrak{g}'/\mathfrak{h}'}^{-1}}{\to} \mathcal{U}(\mathfrak{g}')^{G'}, \tag{1.14}$$

where $\gamma_{\mathfrak{g}/\mathfrak{h}}$ is the Harish-Chandra isomorphism. (see [H-C 1, Theorem 2, p. 125] and [H-C 2, Lemma 13, p. 466])

1. The Classical Invariant Theory in terms of characters.

Let

$$Sp^c = \{(g,\xi);\ g \in Sp,\ det(g-1) \neq 0\}.$$

Then the set

$$\widetilde{Sp^c} = \{(g,\xi);\ g \in Sp^c,\ \xi^2 = det(i(g-1))^{-1}\}$$

is a real analytic manifold, and a two fold covering of Sp^c via the map

$$\widetilde{Sp^c} \ni (g,\xi) \to g \in Sp^c. \tag{2.1}$$

Let

$$\Theta : \widetilde{Sp^c} \ni (g,\xi) \to \xi \in \mathbf{C}, \tag{2.2}$$

Theorem 2.3 (Howe [H2]**).** *The map (2.1) extends to the double covering map* $\widetilde{Sp} \to Sp$*, from the metaplectic group* $\widetilde{Sp}$ *to* Sp*, and* Θ *is the character of one of the two oscillator representations of* $\widetilde{Sp}$*.*

Let $\tilde{G}, \tilde{G}' \subseteq \widetilde{Sp}$ be the preimages of $G, G' \subseteq Sp$. Let Π' be an irreducible admissible representation of $\tilde{G}'$ corresponding via Howe's Correspondence to an irreducible admissible representation Π of $\tilde{G}$, [H1].

Theorem 2.4. *If* G' *is compact, then (in terms of generalized functions)*

(a) $$\int_{\tilde{G}'} \overline{\Theta_{\Pi'}(g')}\, \Theta(g'g)\, dg' = \Theta_{\Pi}(g) \qquad (g \in \tilde{G}),$$

and

(b) $$\int_{\tilde{G}'} \left(\mathcal{C}(z) \cdot \overline{\Theta_{\Pi'}(g')}\right) \Theta(g'g)\, dg' = z \cdot \Theta_{\Pi}(g) \qquad (z \in \mathcal{U}(g)^G),$$

The Cauchy Harish-Chandra Integral:

(2.5) CHC : invariant eigen-distributions on $\tilde{G}' \to$ invariant distributions on $\tilde{G}$,

is a construction, based on limits of holomorphic functions and Harish-Chandra's orbital integrals, valid for an arbitrary dual pair (G', G), with the rank of G' less or equal than the rank of G, see [P2, Def. 2.17]. If G' is compact, then

$$CHC(\overline{\Theta_{\Pi'}})(g) = \int_{\tilde{G}'} \overline{\Theta_{\Pi'}(g')}\, \Theta(g'g)\, dg',$$

as in (2.4.a).

Conjecture 2.6. *With the above notation we have:*

$$CHC \circ \mathcal{C}(z) = z \circ CHC \qquad (z \in \mathcal{U}(g)^G).$$

In our joint work in progress, with Florent Bernon, we have already verified this conjecture for the pairs $(GL_m(\mathbb{D}), GL_m(\mathbb{D}))$, $(O_p(\mathbb{C}), Sp_{2n}(\mathbb{C}))$, $(U_{p,q}, U_{r,s})$ with $p+q = r+s$, and for the pairs (G', G) in deep stable range (see [D - P]).

3. A question.

For a Lie group G with the Lie algebra $\mathfrak{g}$, and the dual $\mathfrak{g}^*$, we may identify the cotangent bundle T^*G with the direct product $G \times \mathfrak{g}^*$, so that for any $f \in C_c^\infty(G)$, df is a $\mathfrak{g}^*$-valued function given by

$$df(g)(x) = \frac{d}{dt} f(g\, exp(tx))|_{t=0} \qquad (g \in G, x \in \mathfrak{g}). \tag{3.1}$$

Under this identification the action by the left translations on T^*G is identified with the left translations on G times the identity on $\mathfrak{g}^*$. The right translations become the right translations on G times the coadjoint action on $\mathfrak{g}^*$. In particular the wave front set, (see [Hö]), of a conjugation invariant distribution on G is a subset of $G \times \mathfrak{g}^*$, invariant under the simultaneous action by conjugation on G and the coadjoint action on $\mathfrak{g}^*$. (See [H3] for more details.)

The following lemma, [P2, Lemma 12.2], is the key to the construction of the Cauchy Harish-Chandra Integral.

Lemma 3.2. *With the above identifications we have*

(a) $$WF(\Theta) = \{(g, \xi) \in \widetilde{Sp} \times sp^*;\ (1, \xi) \in WF(\Theta) \text{ and } Ad^*(g)\xi = \xi\}$$

and

(b) $$\{\xi \in sp^*;\ (1, \xi) \in WF(\Theta)\}$$
is one of the two non-zero minimal nilpotent coadjoint orbits in sp^*.

Thus, it would be interesting to know, for which groups and for which characters does the formula (3.2.a) hold. Since the notion of the wave front set for p-adic groups was rigorously defined in [He], (see also [P1]), this question makes sense and seems interesting even for those groups.

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