An inclusion of orbits sets and surjectivity of the restriction map of algebras of invariants

(T. Ohta)

§0. Problem

$$\begin{array}{lcccl} \text{an alg. group} & \tilde{G} & \curvearrowright & \tilde{X} & : \text{an affine variety} \\ & \cup\!\!\!\!\;| & & \cup\!\!\!\!\;| & \\ \text{a closed subgroup} & G & \curvearrowright & X & : \text{a closed subvariety} \end{array}$$

<u>Inc</u> The map

$$\begin{array}{ccc} X/G & \longrightarrow & \tilde{X}/\tilde{G} \\ \in & & \in \\ \mathcal{O} & \longmapsto & \tilde{\mathcal{O}} := \tilde{G}\cdot\mathcal{O} \end{array}$$

is injective $\square$

<u>Surj</u> The restriction map

$$\begin{array}{cccc} \mathrm{rest}: & \mathbb{C}[\tilde{X}]^{\tilde{G}} & \longrightarrow & \mathbb{C}[X]^{G} \\ & \in & & \in \\ & f & \longmapsto & f|_X \end{array}$$

is surjective. $\square$

Question "(Inc) $\Rightarrow$ (Surj')" ?

The answer is "No" in general.

Today's talk

$$\begin{array}{l} G \subset \tilde{G} \subset GL(V) \\ \quad\quad\quad\quad\quad\quad \circlearrowright \mathrm{Ad} \\ X \subset \tilde{X} \subset \mathrm{End}(V) \end{array} \ : \text{subspaces}$$

I give a condition for which (Inc) and (Surj') hold.

Notation

(1) $X^{G\text{-cl}} := \{ x \in X \mid G \cdot x \text{ is closed in } X \}$

(2) For $\mathcal{O} \in X/G$,

$\mathcal{O}^{G\text{-cl}} :=$ [the unique closed G-orbits in $\overline{\mathcal{O}}$] $\in X^{G\text{-cl}}/G$

Fact 1 $\quad X^{G\text{-}cl}/G \simeq \mathrm{Spec}(\mathbb{C}[X]^G) = X/\!/G \qquad \square$

Fact 2 $\quad \mathrm{rest} : \mathbb{C}[\tilde{X}]^{\tilde{G}} \longrightarrow \mathbb{C}[X]^G$

$\Big\downarrow$ corresponding morphism

$$\begin{array}{ccc} r : \mathrm{Spec}(\mathbb{C}[X]^G) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[\tilde{X}]^{\tilde{G}}) \\ \| & & \| \\ X^{G\text{-}cl}/G & & \tilde{X}^{\tilde{G}\text{-}cl}/\tilde{G} \\ \Downarrow & & \Downarrow \\ \mathcal{O} & \longmapsto & \tilde{\mathcal{O}}^{\tilde{G}\text{-}cl} \end{array}$$

$$\tilde{\mathcal{O}}^{\tilde{G}\text{-}cl} = \left[\begin{array}{l}\text{the unique closed } \tilde{G}\text{-orbit} \\ \text{in } \overline{(\tilde{\mathcal{O}})} = \overline{\tilde{G}\cdot\mathcal{O}}\end{array}\right]$$

Hence "r is injective $\Rightarrow$ rest is surjective". $\square$

Fact 3 $\quad$ Suppose

(Inc) $\quad X/G \hookrightarrow \tilde{X}/\tilde{G}$, $\mathcal{O} \mapsto \tilde{\mathcal{O}} = \tilde{G}\cdot\mathcal{O}$

(CC) $\quad \mathcal{O} \in X^{G\text{-}cl}/G \Rightarrow \tilde{\mathcal{O}} \in \tilde{X}^{\tilde{G}\text{-}cl}/\tilde{G}$.

Then $\quad \mathrm{rest} : \mathbb{C}[\tilde{X}]^{\tilde{G}} \twoheadrightarrow \mathbb{C}[X]^G \qquad \square$

§ 1. Main theorem

Main theorem

V : a vector space $/\mathbb{C}$, $\dim V < \infty$

$\sigma : \mathrm{End}\,V \to \mathrm{End}\,V$

: $\mathbb{C}$-linear anti-automorphism of ass. alg.

$\tilde{G} \subset GL(V)$: a subgroup s.t.

(a) $\langle \tilde{G} \rangle_{\mathbb{C}} \cap GL(V) = \tilde{G}$

where $\langle \tilde{G} \rangle_{\mathbb{C}} = \{ \sum a_i g_i \mid a_i \in \mathbb{C}, g_i \in \tilde{G} \} \subset \mathrm{End}\,V$.

(b) $\sigma(\tilde{G}) \subset \tilde{G}$ and $\sigma^2|_{\tilde{G}} = id_{\tilde{G}}$

Put $G := \{ g \in \tilde{G} \mid \sigma(g) = g^{-1} \}$

and take $\alpha \in Z_{GL(V)}(\tilde{G})$.

(i) Suppose $X, Y \in \mathrm{End}\,V$ satisfy

$$\sigma(X) = \underset{\substack{\uparrow \\ \text{left action}}}{\alpha} X, \quad \sigma(Y) = \alpha Y$$

Then

$$X \underset{Ad(\tilde{G})}{\sim} Y \Rightarrow X \underset{Ad(G)}{\sim} Y .$$

(ii) $\tilde{L} \subset \operatorname{End} V$: a subspace s.t.

(c) $\sigma(\tilde{L}) = \tilde{L}$ and $Ad(\tilde{G})\tilde{L} = \tilde{L}$

(d) $\alpha \tilde{L} = \tilde{L}$

Put $L := \{X \in \tilde{L} \mid \sigma(X) = \alpha X\}$. Then

$$\begin{array}{ccc} L/G & \hookrightarrow & \tilde{L}/\tilde{G} \quad \cdots\cdots (*) \\ \Psi & & \Psi \\ \mathcal{O} & \longmapsto & \tilde{\mathcal{O}} = \tilde{G}\cdot\mathcal{O} \end{array}$$

(iii) Suppose furthermore,

(e) $\tilde{G}$ is reductive,

(f) $g \in \tilde{G}$, $Ad(g)|_{\tilde{L}} = id_{\tilde{L}}$
$\Rightarrow$ g is a scalar matrix

(g) $\varphi := \left(\alpha^{-1}\circ\sigma : \begin{array}{ccc} \tilde{L} & \to & \tilde{L} \\ \Psi & & \Psi \\ X & \mapsto & \alpha^{-1}\sigma(X) \end{array}\right) \in GL(\tilde{L})$

has finite order (i.e., $\varphi^k = id_{\tilde{L}}$ $\exists k \neq 0$)

Then $L^{G\text{-}ss}/G \hookrightarrow L^{\tilde{G}\text{-}ss}/\tilde{G}$ by $(*)$.

Hence $\mathrm{rest} : \mathbb{C}[\tilde{L}]^{\tilde{G}} \twoheadrightarrow \mathbb{C}[L]^G$. $\square$

(iii) is proved by the following:

<u>Theorem</u> (Luna's criterion)

$$\begin{tikzcd} \text{a red. group } \tilde{H} \arrow[r, phantom, "\curvearrowright"] & X \text{ : an affine variety} \\ \text{a red. subgp. } H \arrow[u, hook] & \end{tikzcd}$$

$$\begin{tikzcd} \tilde{H} \arrow[r, phantom, "\curvearrowright"] & X \\ Z_{\tilde{H}}(H) \arrow[u, hook] \arrow[r, phantom, "\curvearrowright"] & X^H := \{x \in X \mid hx = x \text{ for } {}^\forall h \in H\} \arrow[u, hook] \end{tikzcd}$$

Then, for $x \in X^H$,

$\tilde{H} \cdot x$ is closed in $X \Leftrightarrow Z_{\tilde{H}}(H) \cdot x$ is closed in X^H.

$\square$

Use this by putting

$$\begin{tikzcd} & H := \langle \varphi \rangle & \\ GL(\tilde{L}) \supset & \tilde{H} := \langle Ad_{\tilde{L}}(\tilde{G}) \cup \{\varphi\} \rangle \arrow[u, phantom, "\cap" rotate=180] \arrow[r, phantom, "\curvearrowright"] & X = \tilde{L} \\ & Z_{\tilde{H}}(H) = \langle Ad_{\tilde{L}}(G) \cup \{\varphi\} \rangle \arrow[u, hook] \arrow[r, phantom, "\curvearrowright"] & X^H = L \arrow[u, hook] \end{tikzcd}$$

§2. FFT for GL_n and FFT for O_n, Sp_n

$$V = \mathbb{C}^n \oplus \mathbb{C}^m, \quad K = \begin{cases} 1_n \\ \left(\begin{array}{c|c} 0 & 1_{n/2} \\ \hline -1_{n/2} & 0 \end{array}\right) \end{cases};$$

$$J := \left(\begin{array}{c|c} K & 0 \\ \hline 0 & 1_m \end{array}\right)$$

$$\begin{array}{rccc} \sigma : & \mathrm{End}\, V & \longrightarrow & \mathrm{End}\, V \\ & \Psi & & \Psi \\ & X & \longmapsto & \sigma(X) = J^{-1}\,{}^t X J \end{array}$$

$$\sigma\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} K^{-1}\,{}^t A K & K^{-1}\,{}^t C \\ {}^t B K & {}^t D \end{pmatrix}$$

$$\tilde{G} := \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & c 1_m \end{array}\right) \;\middle|\; g \in GL_n(\mathbb{C}),\ c \in \mathbb{C}^\times \right\}$$

$$\begin{aligned} G &:= \{ g \in \tilde{G} \mid \sigma(g) = g^{-1} \} \\ &= \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \pm 1_m \end{array}\right) \;\middle|\; K^{-1}\,{}^t g K = g^{-1} \right\} = \begin{cases} O_n(\mathbb{C}) \times \{\pm 1\} \\ Sp_n(\mathbb{C}) \times \{\pm 1\} \end{cases} \end{aligned}$$

$$\tilde{\mathcal{L}} := \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \;\middle|\; B \in \mathrm{Mat}_{n,m}(\mathbb{C}),\ C \in \mathrm{Mat}_{m,n}(\mathbb{C}) \right\}$$

$$\cup$$

$$\begin{aligned} \mathcal{L} &:= \{ X \in \tilde{\mathcal{L}} \mid \sigma(X) = X \} \\ &= \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline {}^t B K & 0 \end{array}\right) \;\middle|\; B \in \mathrm{Mat}_{n,m}(\mathbb{C}) \right\} \end{aligned}$$

Since $\mathrm{Ad}_{\tilde{L}}(\tilde{G}) = \mathrm{Ad}_{\tilde{L}}(GL_n(\mathbb{C}))$ and

$$\mathrm{Ad}_{\tilde{L}}(G) = \begin{cases} \mathrm{Ad}_{\tilde{L}}(O_n(\mathbb{C})) \\ \mathrm{Ad}_{\tilde{L}}(Sp_n(\mathbb{C})) \end{cases},$$

we can interplete these actions as

$$\begin{array}{ccll} \tilde{G} \curvearrowright & \tilde{L} = \mathrm{Mat}_{m,n} \times \mathrm{Mat}_{n,m} & & \leftarrow \text{by } g\cdot(C,B) = (Cg^{-1}, gB) \\ & \quad\quad\quad ({}^tBK, B) & & \\ \uparrow \qquad & \uparrow \qquad\qquad\quad \uparrow & & \\ G \curvearrowright & L = \mathrm{Mat}_{n,m} \ni B & & \leftarrow \text{by } g\cdot B = gB . \end{array}$$

By the main theorem, we have

<u>Theorem 1</u> (i) $L/G \hookrightarrow \tilde{L}/\tilde{G}$ and any closed G-orbit is maped to a closed $\tilde{G}$-orbit.

(ii) $\mathrm{rest} : \mathbb{C}[\tilde{L}]^{\tilde{G}} \twoheadrightarrow \mathbb{C}[L]^{G}$. $\square$

Define $\pi_{ij} \in \mathbb{C}[\tilde{L}]$ and $\bar{\pi}_{ij} \in \mathbb{C}[L]$ by

$$CB = (\pi_{ij}(C,B)) \in \mathrm{Mat}_{m,m} \quad \text{for } (C,B) \in \tilde{L}$$

matrix entries

and

$${}^tBKB = (\bar{\pi}_{ij}(B)) \in \mathrm{Mat}_{m,m} \quad \text{for } B \in L = \mathrm{Mat}_{n,m}.$$

Then $\bar{\pi}_{ij} = \pi_{ij}|_L$.

<u>FFT for GL_n</u>

$$\mathbb{C}[\tilde{L}]^{\tilde{G}} = \mathbb{C}[\pi_{ij}]$$

$\Downarrow$ Th 1, (ii)

<u>FFT for O_n and Sp_n</u>

$$\mathbb{C}[L]^G = \mathbb{C}[\bar{\pi}_{ij}]$$

§3. Orbits and invariants of classical $\mathbb{Z}_m$-graded Lie algebras

(3.1) θ-representations

G : a reductive algebraic group

$\theta : G \to G$: an automorphism s.t. $\theta^m = id$

$\rightsquigarrow$

$\theta : \mathfrak{g} \to \mathfrak{g}$: autom. of its Lie alg.

$$\zeta := e^{\frac{2\pi i}{m}}$$

$$G_0 := G^\theta = \{ g \in G \mid \theta(g) = g \}$$

Ad $\circlearrowleft$ $\mathfrak{g}_k := \{ X \in \mathfrak{g} \mid \theta(X) = \zeta^k X \}$ $\quad (k \in \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z})$

$(G_0, \mathfrak{g}_1)$: a θ-representation

(3.2) θ-representation of type (AI)

V : a vector space$/\mathbb{C}$, $\dim V < \infty$

$s \in GL(V)$ s.t. $s^m = id_V$

$G := GL(V)$

$$\begin{array}{rccc} \theta : & G & \to & G \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & g & \mapsto & sgs^{-1} \end{array}$$

$\rightsquigarrow$

$(G_0, \mathfrak{g}_1)$: θ-rep. of type (AI)

$V^j := \{ v \in V \mid sv = \zeta^j v \}$ $(j \in \mathbb{Z}_m)$

$G_0 = \{ g \in G \mid gV^j = V^j \ (\forall j) \}$

$\mathfrak{g}_1 = \{ X \in \mathfrak{g} \mid XV^j \subset V^{j+1} \ (\forall j) \}$

<u>Proposition 2</u> (Classification of orbits)

(i) For $A \in \mathfrak{g}_1$, there exists a direct sum decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

of V into subspaces s.t.

(a) Each V_j is A-stable and s-stable.

(b) V_j is indecomposable in the sense of (a).

(ii) Suppose $A|_{V_j}$ has eigenvalue 0.

Then $A|_{V_j}$ is nilpotent and there exists a basis $\{u_i\}_{i=0}^{r}$ of V_j such that

$$\begin{array}{ccccccccccc}
u_0 & \xrightarrow{A} & u_1 & \xrightarrow{A} & u_2 & \xrightarrow{A} & \cdots & \xrightarrow{A} & u_r & \xrightarrow{A} & 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
V^{k} & & V^{k+1} & & V^{k+2} & & & & V^{k+r} & &
\end{array}$$

$$\Downarrow$$

$$\Delta(A|_{V_j}) = \begin{array}{|c|c|c|c|c|}\hline 3^{k} & 3^{k+1} & 3^{k+2} & \cdots\cdots & 3^{k+r} \\ \hline \end{array} \ : \text{a diagram}$$

$$= \Delta(0, 3^{k}, r)$$

(iii) Suppose that $A|_{V_j}$ has an eigenvalue $\alpha \neq 0$.
Then

$$V_j \simeq \bigoplus_{i \in \mathbb{Z}_m} \mathbb{C}[T]/((T-\zeta^i\alpha)^{r+1}) \quad (\exists r \geqslant 0)$$

as $\mathbb{C}[T]$-module. $\left(\begin{array}{l} \mathbb{C}[T] \longrightarrow \mathrm{End}\, V \\ \quad T \longmapsto A \end{array}\right)$

In this case, we put

$$\Delta(A|_{V_j}) = \Delta(\langle \zeta \rangle \cdot \alpha, r)$$

$$= \left(\begin{array}{|c|c|c|c|c|} \hline 1 & \zeta & \zeta^2 & \cdots & \zeta^r \\ \hline \zeta & \zeta^2 & \zeta^3 & \cdots & \zeta^{r+1} \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \hline \zeta^{m-1} & \zeta^m & \zeta^{m+1} & \cdots & \zeta^{m+r-1} \\ \hline \end{array} \;, \langle \zeta \rangle \cdot \alpha \right)$$

: a diagram with eigenvalues.

(iv) Put

$$\Delta(A) := \sum_{j=1}^{l} \Delta(A|_{V_j})$$

We call such a sum a $\langle \zeta \rangle$-signed Young diagram with eigenvalues.

Then the correspondence $A \mapsto \Delta(A)$ is well-defined and $\Delta(A)$ depends only on the orbits $G_0 \cdot A$. Moreover

$$
\begin{array}{ccc}
\mathfrak{g}_1/G_0 & \rightarrow & \left\{ \Delta \;\middle|\; \begin{array}{l} \Delta \text{ is a } \langle 3 \rangle\text{-signed Y.D.} \\ \text{with eigenvalues such that} \\ (\text{the number of } 3^{k}\text{'s in } \Delta) = \dim V^{k}(\text{[illegible]}) \end{array} \right\} \\
\Downarrow & & \Downarrow \\
G_0 \cdot A & \longmapsto & \Delta(A)
\end{array}
$$

is a bijection. $\square$

In particular, semisimple G_0-orbits in $\mathfrak{g}_1$ are determined by eigenvalues with multiplicity, We have:

<u>Theorem 3</u>

(i)
$$
\begin{array}{ccccc}
 & & \mathrm{rest} & & \mathfrak{gl}(V)^{GL(V)\text{-}ce} \\
 & & \Updownarrow & & \| \\
\mathfrak{g}_1^{G_0\text{-}ce}/G_0 = \mathfrak{g}_1^{ss}/G_0 & & \xhookrightarrow{\;r\;} & & \mathfrak{gl}(V)^{s.s}/GL(V) \\
\Downarrow & & & & \Downarrow \\
\mathcal{O} & & \longmapsto & & \widetilde{\mathcal{O}}
\end{array}
$$

(ii) $\mathrm{rest} : \mathbb{C}[\mathfrak{gl}(V)]^{GL(V)} \twoheadrightarrow \mathbb{C}[\mathfrak{g}_1]^{G_0}$

(3.3) θ-representations of O_n and Sp_n.

$(\ ,\ ): V \times V \to \mathbb{C}$: non-deg. ε-form ($\varepsilon = \pm 1$).

$$\begin{array}{ccc} \sigma : \mathrm{End}\,V & \to & \mathrm{End}\,V \\ \Downarrow & & \Downarrow \\ X & \longmapsto & X^* \end{array} \quad : \text{adjoint w.r.to } (\ ,\ ).$$

Put $\tilde{G} = GL(V)$ and

$$G := \{ g \in \tilde{G} \mid \sigma(g) = g^{-1} \} \simeq \begin{cases} O(V) & (\varepsilon = 1) \\ Sp(V) & (\varepsilon = -1) \end{cases}.$$

<u>Fact</u> Let $\theta : G \to G$ be an automorphism of finite order, except for the graph automorphism of order 3. Then

$\exists\, s \in G$ s.t. $s^m = id_V$ and $\theta(g) = sgs^{-1}$. $\square$

Thus we obtain 2 θ-representations:

$$\begin{array}{ccc} (G, \theta) & & (\tilde{G}, \theta) \\ \downarrow & & \downarrow \\ (G_0, \mathfrak{g}_1) & \hookrightarrow & (\tilde{G}_0, \tilde{\mathfrak{g}}_1) \end{array},$$

$\mathfrak{g}_1 = \{ X \in \tilde{\mathfrak{g}}_1 \mid \sigma(X) = -X \}$

Theorem. 4

(i) $\quad \mathfrak{g}_1/G_0 \hookrightarrow \tilde{\mathfrak{g}}_1/\tilde{G}_0 = \left\{ \begin{array}{l} \langle 3\rangle\text{-signed Y.D} \\ \text{with e-values} \end{array} \right\}$

(ii)

$$\begin{tikzcd} \mathbb{C}[\mathfrak{gl}(V)]^{GL(V)} \arrow[r, two heads, "\text{rest}"] \arrow[d, "\text{rest}"'] & \mathbb{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0} \arrow[d, "\text{rest}"] \\ \mathbb{C}[\mathfrak{g}]^{G} \arrow[r, two heads, "\text{rest}"] & \mathbb{C}[\mathfrak{g}_1]^{G_0} \end{tikzcd}$$

(3.4) θ-representation of type (AO)

$\theta : GL(V) \to GL(V)$: outer autom. of finite order.

$^{\exists}\sigma : \operatorname{End} V \to \operatorname{End} V$: anti-autom. s.t.

- $\sigma^{2m} = id$
- $\theta(g) = \sigma(g)^{-1} \quad (g \in GL(V))$

Then

$$\Theta : \begin{array}{ccc} GL(V) & \longrightarrow & GL(V) \\ \Downarrow & & \Downarrow \\ g & \longmapsto & \sigma^2(g) \end{array} \quad : \text{inner autom. of order } m,$$

$^{\exists}S \in GL(V)$ s.t.

- $\Theta(g) = S g S^{-1}$
- $S^m = id_V$

Then we obtain two θ-rep's

$$\begin{array}{ccc} \theta : GL(V) \to GL(V), & & \Theta : \begin{array}{ccc} GL(V) & \to & GL(V) \\ \Downarrow & & \Downarrow \\ g & \longmapsto & \theta^2(g) = SgS^{-1} \end{array} \\ \Downarrow & & \Downarrow \\ (G_0, \mathfrak{g}_1) & \longleftrightarrow & (\tilde{G}_0, \tilde{\mathfrak{g}}_1) \end{array}$$

Put $\zeta = e^{\frac{2\pi i}{2m}}$. We see

$$G_0 = \{ g \in \tilde{G}_0 \mid \sigma(g) = g^{-1} \}$$

$$\tilde{\mathfrak{g}}_1 = \{ X \in \mathfrak{gl}(V) \mid \sigma^2(X) = \theta(X) = \zeta^2 X \}$$

$$\cup$$

$$\mathfrak{g}_1 = \{ X \in \tilde{\mathfrak{g}}_1 \mid -\sigma(X) = \theta(X) = \zeta X \}.$$

Therefore

Theorem 5

(i) $\mathfrak{g}_1 / G_0 \hookrightarrow \tilde{\mathfrak{g}}_1 / \tilde{G}_0 = \left\{ \begin{array}{l} \langle \zeta^2 \rangle\text{-signed Y.D.} \\ \text{with e-values} \end{array} \right\}$

(ii) $\mathbb{C}[\mathfrak{gl}(V)]^{GL(V)} \xrightarrow{\text{rest}} \mathbb{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0} \xrightarrow{\text{rest}} \mathbb{C}[\mathfrak{g}_1]^{G_0}$ $\square$

(3.5) Invariants of Weyl groups

$\theta : G \to G$, $G < GL(n, \mathbb{C})$

$\downarrow$

$(G_0, \mathfrak{g}_1)$: a classical θ-representation as above.

$\mathfrak{a} < \mathfrak{g}_1$: a Cartan subspace

$W := N_{G_0}(\mathfrak{a}) / Z_{G_0}(\mathfrak{a})$: the Weyl group of $(G_0, \mathfrak{g}_1)$

$\uparrow$

complex reflection group

From the above discussion and general theory

$$\mathbb{C}[\mathfrak{gl}(n,\mathbb{C})]^{GL(n,\mathbb{C})} \xrightarrow{\text{rest}} \mathbb{C}[\mathfrak{g}_1]^{G_0} \underset{\uparrow}{\overset{\text{rest}}{\xrightarrow{\sim}}} \mathbb{C}[\mathfrak{a}]^W$$

Vinberg

Hence

$$\mathbb{C}[\mathfrak{a}]^W = \mathbb{C}[\mathfrak{gl}(n,\mathbb{C})]^{GL(n,\mathbb{C})} \Big|_{\mathfrak{a}}$$

§4. actions $(g,h)\cdot(X,Y) = (gXh^{-1}, gYh^{-1})$

and $\quad g\cdot X = g X {}^t g$.

$$K := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \quad J := \left(\begin{array}{c|c} 0 & K \\ \hline K & 0 \end{array}\right)$$

$$\sigma : \mathfrak{gl}(4n,\mathbb{C}) \to \mathfrak{gl}(4n,\mathbb{C})$$
$$\quad X \longmapsto J^{-1}\,{}^tX\,J$$

$$\tilde{G} := \left\{ \left(\begin{array}{cc|cc} g & 0 & & \\ 0 & g & \multicolumn{2}{c}{0} \\ \hline & & h & 0 \\ \multicolumn{2}{c|}{0} & 0 & h \end{array}\right) \,\middle|\, g,h \in GL_n \right\}, \quad \sigma\begin{pmatrix} g & & & \\ & g & & \\ & & h & \\ & & & h \end{pmatrix} = \begin{pmatrix} {}^th & & & \\ & {}^th & & \\ & & {}^tg & \\ & & & {}^tg \end{pmatrix}$$

$$G := \{ g \in \tilde{G} \mid \sigma(g) = g^{-1} \} = \left\{ \left(\begin{array}{cc|cc} g & 0 & & \\ 0 & g & \multicolumn{2}{c}{0} \\ \hline & & {}^tg^{-1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & {}^tg^{-1} \end{array}\right) \,\middle|\, g \in GL(n,\mathbb{C}) \right\}$$

$$\tilde{L} := \left\{ \left(\begin{array}{c|cc} & X & 0 \\ 0 & 0 & Y \\ \hline 0 & \multicolumn{2}{c}{0} \end{array}\right) \,\middle|\, X, Y \in \mathfrak{gl}(n,\mathbb{C}) \right\}$$

$$\sigma\left(\begin{array}{c|cc} & X & 0 \\ 0 & 0 & Y \\ \hline 0 & \multicolumn{2}{c}{0} \end{array}\right) = \left(\begin{array}{c|cc} & {}^tY & 0 \\ 0 & 0 & {}^tX \\ \hline 0 & \multicolumn{2}{c}{} \end{array}\right)$$

$$L := \{ X \in \tilde{L} \mid \sigma(X) = X \} = \left\{ \left(\begin{array}{c|cc} & X & 0 \\ 0 & 0 & {}^tX \\ \hline 0 & \multicolumn{2}{c}{0} \end{array}\right) \,\middle|\, g \in \mathfrak{gl}(n,\mathbb{C}) \right\}$$

Then we can see these actions as

$$\begin{tikzcd} (g, {}^t g^{-1}) \in & \tilde{G} = GL(n,\mathbb{C})^2 \arrow[r, phantom, "\overset{①}{\curvearrowright}"] & \tilde{L} = \mathfrak{gl}(n,\mathbb{C})^2 & \ni (X, {}^tX) \\ g \arrow[u, mapsto] \in & G = GL(n,\mathbb{C}) \arrow[u, hook] \arrow[r, phantom, "\overset{②}{\curvearrowright}"] & L = \mathfrak{gl}(n,\mathbb{C}) \arrow[u, hook] & \ni X \arrow[u, mapsto] \end{tikzcd}$$

where

① $(g,h)\cdot(X,Y) = (gXh^{-1}, gYh^{-1})$

② $g\cdot X = gX\,{}^tg$

Hence

<u>Prop 6</u> $\quad L/G \hookrightarrow \tilde{L}/\tilde{G}$ $\qquad\square$

<u>Remark</u> In this case,

$$\mathbb{C}[\tilde{L}]^{\tilde{G}} = \mathbb{C} = \mathbb{C}[L]^G,$$

but we can verify

$$\mathbb{C}(\tilde{L})^{\tilde{G}}\big|_L = \mathbb{C}(L)^G \qquad \square$$