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Singular invariant eigendistributions associated to symmetric spaces

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$G \supset H$ symmetric pair

Understand $H \backslash G / H$ orbits classified (Matsuki-Oshima)

$$C^{-\infty}(H \backslash G/H) = C^{-\infty}(G)^{H \times H}$$

§ generalized Gelfand pair.

§ tangent space (isotropic cases)

§ ——— " ——— (non-isotropic cases)

§ global case.

§ remarks

§1 G : connected Lie group.
$\cup$
K : compact

(G, K) is called Gelfand pair.

$\overset{\text{def}}{\iff}$ $L^1(K \backslash G / K)$: commutative (under convolution)

$\iff$ $\mathbb{D}(G/K)$: commutative
$\quad$ G-inv. diff. operator

$\Leftrightarrow$ $(\pi, \mathcal{H})$: irreducible unitary reprn of G, then

$\dim \mathcal{H}^{-\infty, K} \leq 1$

<u>Example</u> G/K : Riemannian symm. space.

$\Rightarrow$ Gelfand pair.

spherical homogeneous space.

<u>e.g.</u> $SU(2)^3 / \underset{\text{diag}}{} SU(2)$

$(SU(n+1) \times SU(n)) / \text{diag}\, SU(n)$

$K \neq$ compact. $G \supset H$: closed

(G, H) : gen. Gelfand pair

$\overset{\text{def}}{\Longleftrightarrow}$ $(\pi, \mathcal{H})$: irred. unitary repr. of G then

$\dim \mathcal{H}^{-\infty, H} \leq 1$

<u>Example (van Dijk)</u>

rank one semisimple symm. space G/H

$\begin{cases} G/H \neq SO(1,n)/SO(1,n-1) \Rightarrow \text{gen } G. \text{ pair.} \\ SO(1,n)/SO(1,n-1) \Rightarrow \text{NOT gen. } G. \text{ pair.} \end{cases}$

<u>Rmk</u> $SO_0(1,n)/SO_0(1,n-1)$: G-pair.

"J-criterion" (Thomas '84)

τ: involution of G with $\tau(H) = H$.

eigendistribution of $\mathbb{D}(G/H)$

ここにもある.

$\lfloor$ any $\varphi \in C^{-\infty}(G)^{H\times H}$ satisfies $\varphi(\tau(g^{-1})) = \varphi(g)$

$\Rightarrow$ (G, H) : gen. G. pair.

Note: "J-criterion on orbits":

$g \in G$ satisfies $\tau(g) \in H g^{-1} H$ $\rfloor$

J-criterion on orbits (B) $\not\Longrightarrow$ J-criterion for gen G. pair. (A)

not yet proved.

(A) $\not\Leftarrow$ (B) $\exists$ counter example

§2 $G = SO_0(p+1, q) \supset H = SO_0(p, q)$

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ B : Killing form on $\mathfrak{g}$

σ 1 -1 $Q(x) = B(x, x)$ $x \in \mathfrak{q}$.

(A) $\Rightarrow$ (B) は
inv. distribution separates orbit と同じ

$C^{-\infty}(G)^{H\times H} = C^{-\infty}(G/H)^{H}$ left.

$C^{-\infty}(\mathfrak{q})^{Ad(H)}$

$Q \in \mathbb{R}[\mathfrak{q}]^H$: generator ($\because$ G/H is rank $= 1$)

~~$Q : \mathfrak{q} \to \ldots$~~ $Q : \mathfrak{q} \setminus \{0\} \longrightarrow \mathbb{R}$

then the level set $Q^{-1}(c)$ is an H-orbit.

(isotropic symmetric space)

orbit classifying map.

No. 4

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$\{\varphi\circ Q \mid \varphi \in C^{-\infty}(\mathbb{R})\}$

$\|$

on $\mathfrak{g}\setminus\{0\}$ $\quad C^{-\infty}(\mathfrak{g}\setminus\{0\})^H = Q^*(C^{-\infty}(\mathbb{R}))$

B $\Longrightarrow$ A on "$\mathfrak{g}\setminus\{0\}$"

$$0 \to \Gamma_0 C^{-\infty}(\mathfrak{g}) \to C^{-\infty}(\mathfrak{g}) \longrightarrow C^{-\infty}(\mathfrak{g}\setminus\{0\}) \longrightarrow 0$$

$$0 \to \Gamma_0(C^{-\infty}(\mathfrak{g}))^H \to C^{-\infty}(\mathfrak{g})^H \to \underbrace{C^{-\infty}(\mathfrak{g}\setminus\{0\})^H}_{\ni\,\varphi} \to 0$$

If φ has an extension on $\mathfrak{g}$ as $C^{-\infty}$, then we have an H-inv. ext. on $\mathfrak{g}$ as $C^{-\infty}$

(pf. uses Lie alg cohomology $H^1(\mathfrak{g}, *)=0$) $\quad (p,q)\neq(1,1)$

§3 $\quad G = SL(n+1,\mathbb{R}) \supset H = GL(n,\mathbb{R})$ or $GL(n,\mathbb{R})_+$

$\mathfrak{g} = \mathbb{R}^n \oplus (\mathbb{R}^n)^* \ni (x, y)$

$\quad\quad\quad\quad\quad\quad\quad\quad \mathrm{I}\, h$

$\quad\quad\quad\quad\quad\quad (hx, {}^t h^{-1} y)$

B. $\quad Q = B|_{\mathfrak{g}} \quad Q(x,y) = {}^t y \cdot x$

$\mathfrak{g}_{reg} = \{(x,y) \mid x\neq 0,\ y\neq 0\}$

Orbit classifying

$Q : \mathfrak{g}_{reg} \longrightarrow \mathbb{R}$. & $Q^{-1}(c)$ is an H-orbit

$Q^{-1}(0) = 4$-orbits

$$Q^{-1}(0) = \{(x,y) \mid x \neq 0, y \neq 0, Q(x,y) = 0\}$$

$$\cancel{= \{(x,y) \mid x\neq 0, y \neq 0, Q\}}$$

$$\cup \{(x,0) \mid x \neq 0\} \cup \{(0,y) \mid y \neq 0\} \cup \{(0,0)\}.$$

<u>Theorem</u> (van Dijk '86, O.–) $\quad n \geq 3$

$\varphi \in C^{-\infty}(\mathfrak{g})^H$ then φ is $SO_0(Q)$-invariant

where $O(Q) = \{g \in GL(\mathfrak{g}) \mid Q(g\xi) = Q(\xi) \ \forall \xi \in \mathfrak{g}\}$

$\simeq O(n,n) \qquad \cup$

$\qquad\qquad\qquad H.$

$SO_0(Q) \cong SO_0(n,n)$

three orbits are merged under the action of $SO_0(Q)$.

$$\left\{\begin{array}{l} G_{ij} = -x_j \dfrac{\partial}{\partial y_i} + x_i \dfrac{\partial}{\partial y_j} = -G_{ji} \\ F_{ij} = -x_j \dfrac{\partial}{\partial x_i} + y_i \dfrac{\partial}{\partial y_j} \end{array}\right) \in \mathrm{Lie}(O(Q))$$

$F_{ij} \in \mathrm{Lie}\, H.$

(*) $G_{ij} = -G_{jk} F_{ki} - G_{ki} F_{kj} - G_{ij} F_{kk}$

$\varphi \in C^{-\infty}(\mathfrak{g})^H \Rightarrow F_{ij}\varphi = 0 \quad \forall i,j$

$\underset{(*)}{\Longrightarrow} G_{ij}\varphi = 0 \Longrightarrow \varphi \in C^{-\infty}(\mathfrak{g})^{SO_0(Q)}$

$\boxed{n=2}$ $\quad H = GL_+(2,\mathbb{R})$

$C^{-\infty}(\mathfrak{g})^H = C^{-\infty}(\mathfrak{g})^{SO_0(Q)} \oplus \mathbb{C}\varphi.$

$\exists! \ \varphi \in C^{-\infty}(\mathfrak{g})^H$ s.t. $\square \varphi = 0$

φ : homog. degree $= -2$

$\mathrm{supp}\, \varphi \subset \overline{N_+}$

$N_\pm = \{(x,y)\in\mathbb{R}^4 \mid Q(x,y)=0,\ x_1y_2-x_2y_1 \gtrless 0\}$

"J-criterion"

$\tau(X) = \mathrm{Ad}\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix}(-{}^tX)$ (n=2)

$(x,y)\longmapsto(y_1,-y_2,x_1,-x_2)$

$\varphi\in C^{-\infty}(\mathfrak{g})^H$ is invariant under $-\tau$

$\boxed{n\geqq 3}$ $\tau(X)=-{}^tX \Longrightarrow$ J-criterion

§ global $G=SL_{n+1}(\mathbb{R})\supset H=GL_n(\mathbb{R})$

$Q: G\longrightarrow\mathbb{R}$ is $Q(g)=1-g_{11}\cdot(g^{-1})_{11}$

$g=\begin{pmatrix}g_{11}&\\&\end{pmatrix}$ $\quad H=\begin{array}{c|cc} & 1 & n\\\hline 1 & (\det h)^{-1} & \\ n & & h\end{array}$

Fact $Q^{-1}(c)$ is $H\times H$-orbit for $c\neq 0,1$

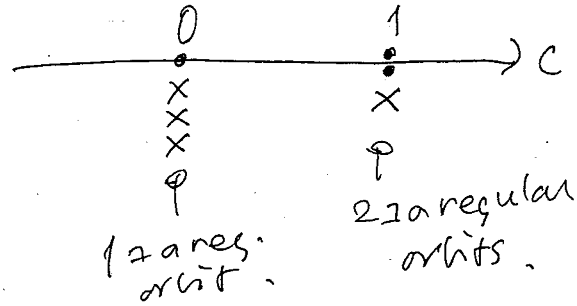
$Q^{-1}(0)\subset G$ is the same as $Q^{-1}(0)\subset\mathfrak{g}$.

$Q^{-1}(1)=\{g_{11}=0,\ (g^{-1})_{11}\neq 0\}$ ← τ-inv.

$\cup\{\quad\neq 0 \qquad =0\}$ ←

$\cup\{\quad =0 \qquad =0\}$ ←

$\tau(g)={}^tg^{-1}$



$$C^{-\infty}(Q^{-1}(0))^{H\times H} \ni \varphi \Rightarrow \varphi(\tau(g^{-1})) = \varphi(g).$$

Eg

$O(Q) = O(n,n)$ $\quad G = SL(n+1,\mathbb{R})$

$H = GL(n,\mathbb{R})_+$

G/H_{ss} $\quad H_{ss} = SL(n,\mathbb{R})$

$\downarrow GL(1)$

G/H

$$G/H_{ss} \xrightarrow{\sim} O(n+1,n+1)/O(n,n+1) \quad \cancel{O(Q)/O(Q')}$$

$(2n+1)$-dim'l $\quad \cancel{\operatorname{sgn} Q = (n,n)}$

$\cancel{\operatorname{sgn} Q' = (n-1,n)}$

$$C^{-\infty}(G/H)^H \simeq C^{-\infty}(\underbrace{SO_0(n,n)}_{= SO_0(Q)}\times SO_0(1,1)\backslash SO_0(n+1,n+1)/SO_0(n,n+1))$$