Resolution of nilpotent orbits for symmetric pairs

Kyo Nishiyama (Kyoto University)

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1. Equivariant double fibration

G, G': reductive linear algebraic groups $/\mathbb{C}$
W: finite dim representation of $G \times G'$

$$\Longrightarrow \begin{cases} Y := W/\!/G = \mathrm{Spec}\,\mathbb{C}[W]^G \\ X := W/\!/G' = \mathrm{Spec}\,\mathbb{C}[W]^{G'} \end{cases}$$

: affine quotients of W,

where
$\mathbb{C}[W] =$ (ring of regular functions on W)
$\mathbb{C}[W]^G =$ (G-invariants)

$$\begin{array}{ccccc} & & W & & \\ & \swarrow & & \searrow & \\ G \curvearrowright X & & & & Y \curvearrowleft G' \end{array}$$

$X/G =$ (set of G-orbits in X)
$Y/G' =$ (set of G'-orbits in Y)

Problem 1 What is the relation between X/G and Y/G'?

An answer to Problem 1:

Theorem 2
Under natural **Assumptions** *below, there is an embedding* $\theta : Y/G' \hookrightarrow X/G$

- Description of the embedding:

$W \xrightarrow{\varphi} X$: G-equiv. quotient map $//G'$
$W \xrightarrow{\psi} Y$: G'-equiv. quotient map $//G$

$\forall \mathcal{O}' \subset Y$: G'-orbit, $\exists \mathcal{O} \subset X$: G-orbit
s.t. $\varphi(\psi^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}$

$$\Longrightarrow \qquad \theta : Y/G' \ni \mathcal{O}' \mapsto \mathcal{O} \in X/G$$

θ preserves nilpotency & closure relation

Classical example

$G = GL_n,\ G' = GL_m$
$W = M_{m,n} \oplus M_{n,m} = \mathbb{C}^n \otimes \mathbb{C}^m \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^*$

Assumption: $n \geq 2m$

$$\Longrightarrow \begin{cases} Y = W//G = M_m \\ X = W//G' = \mathrm{Det}_m(M_n) \end{cases}$$

where

$\mathrm{Det}_m(M_n) = \{A \in M_n \mid \mathrm{rank}\, A \leq m\}$

: determinantal variety of rank m

$GL_m \curvearrowright M_m = Y$: Ad action
$GL_n \curvearrowright \mathrm{Det}_m(M_n) = X$: restrict. of Ad action

- Quotient maps

$$(A, B) \in M_{m,n} \oplus M_{n,m}$$
$$\varphi(A, B) = AB \in M_m = Y$$
$$\psi(A, B) = {}^t(BA) \in \mathrm{Det}_m(M_n) = X$$

- Correspondence

$$\theta : M_m / \mathrm{Ad}\, GL_m \to \mathrm{Det}_m(M_n) / \mathrm{Ad}\, GL_n$$
$$\theta(G' \cdot A) = G \cdot \left[\begin{array}{c|c} A & 0 \\ \hline X & 0 \end{array}\right], \; X\text{: generic}$$

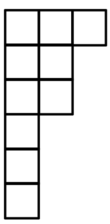
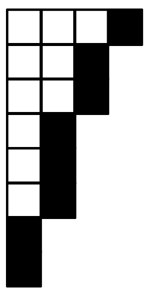
- Extreme cases:

<u>Regular orbits</u>:

$$\mathrm{rank}\, A = m \implies \theta(G' \cdot A) = G \cdot \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right]$$

<u>Nilpotent orbits</u>:

A: nilpotent of type $\lambda = (\lambda_1, \ldots, \lambda_l) \implies$
$\theta(G' \cdot A)$: nilpotent of type $\lambda + (1, \ldots, 1)$
Young diagramatic notaion

$\mathcal{O}' =$  $\Rightarrow$ $\mathcal{O} =$ 

Assumptions

Put
$\mathfrak{N} = \psi^{-1}(\psi(0))$: null cone (for the action of G)

Assumption 1 (a) *The quotient map* $\psi : W \to Y$ *is flat* ($\Leftrightarrow \mathbb{C}[W]$ *is flat* $/\mathbb{C}[W]^G$)

(b) $\exists$ *an open dense* G*-orbit* $\mathcal{O}_0 \subset \mathfrak{N}$

(c) $\mathfrak{N} \simeq W \times_Y \{0\}$ *is reduced*

(d) *Generically, the fiber of* $\varphi : W \to X$ *is a single (hence closed)* G'*-orbit*

(e) *Put* $W^\circ := \coprod$(*closed* G'*-orbits*)
$\Longrightarrow \psi^{-1}(y) \cap W^\circ \neq \emptyset$ ($\forall y \in Y$)

Under the above assumptions (a) – (e), Theorem 2 holds:

$\exists$ embedding $\quad \theta : Y/G' \hookrightarrow X/G$

$\forall \mathcal{O}' \subset Y$: G'-orbit
$\exists \mathcal{O} = \theta(\mathcal{O}')$: G-orbit $\subset X$ s.t.

(1) $\overline{\mathcal{O}} = \varphi(\psi^{-1}(\overline{\mathcal{O}'}))$

$\overline{\mathcal{O}} \simeq (W \times_Y \overline{\mathcal{O}'})//G'$ (as G-variety)

(2) $\mathcal{O}'_1 < \mathcal{O}'_2 \implies \theta(\mathcal{O}'_1) < \theta(\mathcal{O}'_2)$
(closure relation)

(3) $\mathcal{O}'$: nilpot. $\implies \mathcal{O} = \theta(\mathcal{O}')$: nilpot.

Examples of G', G and W

- Tensor product of natural representations

$W = G \otimes G'$	condition
$O_n \otimes O_m$	$2m < n$
$O_n \otimes Sp_{2m}$	$4m < n$
$O_n \otimes SL_m$	$2m < n$
$Sp_{2n} \otimes O_m$	$m \leq n$
$Sp_{2n} \otimes Sp_{2m}$	$2m \leq n$
$Sp_{2n} \otimes SL_m$	$m \leq n$

- Contruction by GL_n, Case (I)

$G = GL_n \curvearrowright V = \mathbb{C}^n$

$W = (V \oplus V^*) \otimes U$

$U = G'$	condition
O_m	$2m \leq n$
Sp_{2m}	$4m \leq n$
SL_m	$2m \leq n$

• Contruction by GL_n, Case (II)

$G = GL_n \curvearrowright V = \mathbb{C}^n$
$G' = G'_+ \times G'_- \curvearrowright U^+ \oplus U^-$
$W = V \otimes U^+ \oplus V^* \otimes U^-$

$U^+ = G'_+$	$U^- = G'_-$	condition
O_p	O_q	$p + q \leq n$
Sp_{2p}	Sp_{2q}	$2p + 2q \leq n$
O_p	Sp_{2q}	$p + 2q \leq n$
SL_p	SL_q	$p + q \leq n$

• Opposite cases

Can exchange the roles of G and G'

• General linear groups

$G = GL_n \curvearrowright V = \mathbb{C}^n$
$G' = GL_m \curvearrowright U = \mathbb{C}^m$
Assume $2m \leq n$
$W = V \otimes U^* \oplus V^* \otimes U$

History

• Case arising from dual pairs: The theorem is primitively noticed by Roger Howe (1970's)

• $G_{\mathbb{C}}$-nilpotent orbits of reductive dual pairs:

Daszkiewicz-Kraśkiewicz-Przebinda (1996, 1997)

• Nilpotent orbits of symmetric pairs related to dual pair:

N. (2000)
Daszkiewicz-Kraśkiewicz-Przebinda (2002)
Ohta (preprint)
N.-Ochiai-Zhu (to appear in TAMS)

• Beyond nilpotent orbits:

Ohta (private communication, 2003)
N. (preprint)

2. Lifting of nilpotent orbits of symmetric pair

• symplectic group

$W = \mathbb{C}^N$: Hermitian inner product $(\,,\,)$
$\langle u, v\rangle = \mathrm{Im}(u, v)$: symplectic form $/\mathbb{R}$
$\Rightarrow W = W_{\mathbb{R}}$: symplectic space $/\mathbb{R}$

$$\begin{aligned} W_{\mathbb{R}} &= \mathbb{R}^N \oplus \sqrt{-1}\mathbb{R}^N \\ &= X \oplus Y : \text{ complete polarization} \end{aligned}$$

symplectic group $Sp(W_{\mathbb{R}})$
$\supset U(\mathbb{C}^N)$: unitary group (max cpt subgrp)

• Dual pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$

$(G_{\mathbb{R}}, G'_{\mathbb{R}})$: subgroups in $Sp(W_{\mathbb{R}})$
mutually commutant to each other in $Sp(W_{\mathbb{R}})$
$K_{\mathbb{R}} \subset G_{\mathbb{R}}, K'_{\mathbb{R}} \subset G'_{\mathbb{R}}$: max compact subgrps
s.t. $K_{\mathbb{R}} \cdot K'_{\mathbb{R}} \subset U(W)$

K, K': $\mathbb{C}$-fication of $K_{\mathbb{R}}, K'_{\mathbb{R}}$

$\Longrightarrow$ $K \times K' \curvearrowright W$: holomorphic action

$\mathfrak{g} = (\operatorname{Lie} G_{\mathbb{R}})_{\mathbb{C}}$: $\mathbb{C}$-fied Lie algebra

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Cartan decomposition

$\mathbb{C}$-fication of moment maps:

$$\begin{cases} \psi : W \to Y = W/\!/K \subset \mathfrak{p}' \\ \varphi : W \to X = W/\!/K' \subset \mathfrak{p} \end{cases}$$

Assume $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ is <u>irreducible</u> and <u>strictly in the stable range</u> ($G'_{\mathbb{R}}$: small) $\Rightarrow$ $K \times K' \curvearrowright W$ satisfies Assumption 1 $Y = \mathfrak{p}'$ $\Rightarrow$ $\exists$embedding $\theta : \mathfrak{p}'/\operatorname{Ad} K' \hookrightarrow \mathfrak{p}/\operatorname{Ad} K$ by Theorem 2

SSR := strictly in the stable range

Example: dual pair $O_{p,q}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})$

SSR $\iff$ $2n < \min\{p, q\}$

(Rmk. $p+q \leq n$ is also SSR in the opposite direction)

- $\mathbb{C}$-fied moment maps:

$$
\begin{array}{ccc}
 & W = W_+ \oplus W_- & \\
 \varphi \swarrow & & \searrow \psi \\
 \mathfrak{p} & & \mathfrak{p}'_+ \oplus \mathfrak{p}'_- = \mathfrak{p}'
\end{array}
$$

$$
\begin{array}{ccc}
 & M_{p,n} \oplus M_{q,n} & \\
 \varphi \swarrow & & \searrow \psi \\
 M_{p,q} & & \mathrm{Sym}_n \oplus \mathrm{Sym}_n
\end{array}
$$

For $(A, B) \in M_{p,n} \oplus M_{q,n}$

$$\varphi(A, B) = A\,{}^tB \in M_{p,q}$$
$$\psi(A, B) = ({}^tAA, {}^tBB) \in \mathrm{Sym}_n \oplus \mathrm{Sym}_n$$

- Lifting of rank k orbit

$K = O_p(\mathbb{C}) \times O_q(\mathbb{C}),\ K' = GL_n(\mathbb{C})$

$\mathcal{O}'_k := \{A \in \mathsf{Sym}_n \mid \operatorname{rank} A = k\} \subset \mathfrak{p}'_+$
: nilpotent $\operatorname{Ad} K'$-orbit

$\mathcal{O}_k = \theta(\mathcal{O}'_k) \subset \mathfrak{p}$: *theta lift* of $\mathcal{O}'_k$
: nilpotent $\operatorname{Ad} K$-orbit

$$\psi^{-1}(\mathcal{O}'_k) \cap W^\circ = \{(A, B) \in M_{p,n} \oplus M_{q,n} \mid \ (*)\ \}$$

$$(*) \begin{cases} \operatorname{rank} {}^t\!AA = k \\ {}^tBB = 0 \\ \operatorname{rank} A = \operatorname{rank} B = n \end{cases}$$

$$\Longrightarrow\ \mathcal{O}_k \simeq (\psi^{-1}(\mathcal{O}'_k) \cap W^\circ)/GL_n$$

• Geometric structure of the orbit $\mathcal{O}_k$

$\mathbb{G}_n(\mathbb{C}^p)$: Grassmannian of n-dim subsp in $\mathbb{C}^p$
$\mathcal{T}_p$: tautological bundle on $\mathbb{G}_n(\mathbb{C}^p)$

$$Z(k;p) := \{V \in \mathbb{G}_n(\mathbb{C}^p) \mid \operatorname{rank} V = k \text{ (as a quadratic space)}\}$$

$$\mathcal{T}(k;p) = \mathcal{T}_p\Big|_{Z(k;p)} : \text{ restriction to } Z(k;p)$$

$\psi^{-1}(\mathcal{O}'_k) \cap W^\circ \hookrightarrow \mathrm{End}(\mathcal{T}(k;p)) \boxtimes \mathrm{End}(\mathcal{T}(0;q)^*)$
(Rmk. $\mathcal{T}(0;q)$ is the tautological bundle on isotropic Grassmannian)

Theorem 3 $\exists$ *$O_p \times O_q$-equiv birational map*

$$\Phi : \mathcal{T}(k;p) \boxtimes \mathcal{T}(0;q)^* \to \overline{\mathcal{O}_k}$$

which gives a rsolution of singularity

- Two extreme cases

$\underline{k=0}$

$\mathcal{O}_0 =$ lifiting of the trivial orbit

$$\iff [(\boxplus\boxminus)^n(\boxminus\boxplus)^n(\boxplus)^{p-2n}(\boxminus)^{q-2n}]$$

$\mathcal{T}(0;p) \boxtimes \mathcal{T}(0;q)^* \to \overline{\mathcal{O}_0}$: resol of singularity

$\mathcal{T}(0;p)$: vector bundle over isotropic Grassmannian

$\underline{k=n}$

$\mathcal{O}_n \iff [(\boxplus\boxminus\boxplus)^n(\boxplus)^{p-2n}(\boxminus)^{q-n}]$

$\mathcal{T}(n;p)$: vector bundle over symmetric space $O_p/O_n \times O_{p-n}$

• Spherical nilpotent orbits for symmetric pair

$\mathcal{O}_0$: spherical nilpotent K-orbit in $\mathfrak{p}$

$\mathcal{P}_n$: partitions of length at most n

$\sigma_\alpha^{(p)}$: irred finite dim representation of O_p
with ht wt $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0) \in \mathcal{P}_n$

$$\mathbb{C}[\overline{\mathcal{O}_0}] \simeq \sum_{\alpha \in \mathcal{P}_n}^{\oplus} \sigma_\alpha^{(p)} \boxtimes \sigma_\alpha^{(q)}$$

$\mathcal{O}_0$ ($n < \min\{p/2, q/2\}$) gives a large part of spherical orbits

Classification of spherical nilpotent
K-orbits in $\mathfrak{p}$ — D. R. King (2004)
(cf. N. (2004) for $U(p,p)$)

3. Unimodular action on bilinear forms

$SL_m(\mathbb{C}) \curvearrowright M_m(\mathbb{C}) = \mathbb{C}^m \otimes \mathbb{C}^m$
via $g \cdot A = gA\,{}^t g \quad (g \in SL_m, A \in M_m)$

(cf. Đoković-Sekiguchi-Zhao (preprint), Ochiai)

Resolution via the action of GL_n

$G = GL_n, G' = SL_m$
$V = \mathbb{C}^n$: natural representation of G
$U = \mathbb{C}^m$: natural representation of G'
$W = (V \oplus V^*) \otimes U = M_{n,m} \oplus M_{n,m}$

$2m \leq n \implies W$ satisfies Assumption 1.

Double fibration by quotient maps:

$$\begin{array}{ccc} & W & \\ & {}^{\varphi}\swarrow \quad \searrow^{\psi} & \\ X = W//G' & & W//G = Y \end{array}$$

$$\begin{cases} X = \mathbb{G}_m^{\text{aff}}(V \oplus V^*) \curvearrowleft GL_n \\ Y = U \otimes U = M_m(\mathbb{C}) \curvearrowleft SL_m \end{cases}$$

$\mathbb{G}_m^{\text{aff}}(\mathcal{V})$: affine cone of Grassmannian in $\bigwedge^m \mathcal{V}$ with Plücker embedding

$\exists$ embedding

$$\theta : M_m/SL_m \hookrightarrow \mathbb{G}_m^{\text{aff}}(V \oplus V^*)/GL(V)$$

θ induces (not necessarily injective) map

$$\theta^\circ : M_m/SL_m \to \mathbb{G}_m(V \oplus V^*)/GL(V)$$

θ° is injective on "nilpotent" orbits

$K_{\mathbb{C}}$-orbits on isotropic Grassmannian

$V \oplus V^*$ carries natural

$$\begin{cases} \text{symplectic bilinear form } \langle , \rangle, \text{ and} \\ \text{symmetric bilinear form } (,) \end{cases}$$

s.t. $V \oplus V^*$ is a complete polarization

$Sp(V \oplus V^*)$, $O(V \oplus V^*)$: isometry groups

$$\Longrightarrow Sp(V \oplus V^*) \cap O(V \oplus V^*) = GL(V)$$

$\mathbb{IG}_m^{Sp} = \mathbb{IG}_m^{Sp}(V \oplus V^*)$, $\mathbb{IG}_m^{O} = \mathbb{IG}_m^{O}(V \oplus V^*)$

: isotropic Grassmannians

Theorem 4 *For* $\mathcal{O}' \in M_m/SL_m$, *put* $\mathcal{O} = \theta^\circ(\mathcal{O}') \in \mathbb{G}_m(V \oplus V^*)/GL(V)$

$$\mathcal{O} \in \mathbb{IG}_m^{Sp}/GL(V) \iff \mathcal{O}' \subset \mathrm{Sym}_m$$

$$\mathcal{O} \in \mathbb{IG}_m^{O}/GL(V) \iff \mathcal{O}' \subset \mathrm{Alt}_m$$

Rmk. $\mathbb{IG}_m^{Sp}/GL(V) \simeq P_m \backslash Sp(V \oplus V^*)/GL(V)$

P_m: a maximal parabolic subgroup in Sp

"$K_{\mathbb{C}}$" $= GL(V)$: $\mathbb{C}$-fied max cpt subgroup

(cf. Matsuki: $\# P \backslash G_{\mathbb{C}}/K_{\mathbb{C}} < \infty$)

Lifting of the trivial orbit

$\mathcal{O}' = \{0\}$: the trivial orbit in $Y = M_m$
$\mathcal{O}^{\mathbf{1}} = \theta(\mathcal{O}')$: lifted orbit in $X = \mathbb{G}_m^{\mathrm{aff}}(V \oplus V^*)$

$$\begin{aligned} \mathfrak{N} &= \psi^{-1}(0) \text{ : null cone} \\ &= \{(A, B) \in M_{n,m} \oplus M_{n,m} \mid {}^t BA = 0\} \\ &\Longrightarrow \overline{\mathcal{O}^{\mathbf{1}}} = \mathfrak{N}//G' \end{aligned}$$

$\mathcal{O}^{\mathbf{1}}$ is a G-spherical variety with the normal closure in $X = \mathbb{G}_m^{\mathrm{aff}}(V \oplus V^*)$

$\mathbb{P}\mathcal{O}^{\mathbf{1}}$: projective image of $\mathcal{O}^{\mathbf{1}}$ in $\mathbb{G}_m$

Theorem 5 $\mathbb{IG}_m^{Sp} \cap \mathbb{IG}_m^{O}$ *is a normal, $GL(V)$-spherical variety which contains $\mathbb{P}\mathcal{O}^{\mathbf{1}}$ as an open dense orbit*

$\mathcal{P}_m$: partitions of length at most m

$\tau_\alpha^{(m)}$: irred finite dim representation of GL_m with ht wt $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{P}_m$

$$\alpha \odot \beta = (\alpha, 0, \dots, 0, \beta^*) \in \mathbb{Z}^n \quad (\alpha, \beta \in \mathcal{P}_m)$$
$$\beta^* = (-\beta_m, -\beta_{m-1}, \dots, -\beta_1)$$

(need the assumption $2m \leq n$)

• Decomposition of the regular function ring $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}}]$ as a representation of $GL_n = GL(V)$

$$\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}}] \simeq \mathbb{C}[\mathfrak{N}]^{G'} \qquad (G' = SL_m)$$
$$\simeq \sum_{k \geq 0}^{\oplus} \sum_{\alpha - \beta^* = k\mathbf{1}_m}^{\oplus} \tau_{\alpha \odot \beta}^{(n)} \qquad (\alpha, \beta \in \mathcal{P}_m)$$

$\Longrightarrow$ $\overline{\mathcal{O}^{\mathbf{1}}}$ is G-spherical
$\Longrightarrow$ so is $\mathcal{O}^{\mathbf{1}}$ and $\mathbb{P}\mathcal{O}^{\mathbf{1}}$

Rmk. $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}}] =$ (homog coord ring of $\mathbb{IG}_m^{Sp} \cap \mathbb{IG}_m^{O}$)

Open Problems

(1) Find a pair (G, G') of exceptional type and a repr W satisfying Assumption 1

(2) V, U: irred reprs of G, G'
Classify (V, U) for which $W = V \otimes U$ satisfies Assumption 1.

(3) Give description of $\theta : Y/G' \hookrightarrow X/G$ in a combinatorial way.

(4) Find a representation theoretic interpretation of θ
(cf. theta correspondence by Howe; Daszkiewicz-Przebinda (1996), N.-Zhu (preprint))

(5) Find the relation between the singularity of $\overline{\mathcal{O}'}$ and that of $\overline{\theta(\mathcal{O}')}$