Homomorphisms between Specht modules and Weyl modules

Andrew Mathas

(Joint work with Sinéad Lyle)

A.Mathas@maths.usyd.edu.au

University of Sydney

The Hecke algebra

Fix a ring R and an integer $n \geq 1$.

Let q be an *invertible* element of R.

The Iwahori–Hecke algebra $\mathscr{H}_n = \mathscr{H}_q(\mathfrak{S}_n)$ is the unital associative R algebra with generators

$$T_1, \ldots, T_{n-1}$$

together with the relations

$$(T_j + q)(T_j - 1) = 0$$

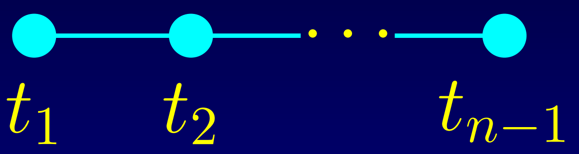
and the braid relations

$$T_{j+1}T_jT_{j+1} = T_jT_{j+1}T_j$$

$$T_jT_k = T_kT_j, \qquad \text{for } |j-k| > 1$$

...first meaning of the definition

The symmetric group $\mathfrak{S}_n$ has presentation given by the diagram



So, $\mathfrak{S}_n = \langle\, t_1, \ldots, t_{n-1} \,\rangle$ with relations:

$$t_i^2 = 1,$$
$$t_j t_{j+1} t_j = t_{j+1} t_j t_{j+1}$$

and $t_j t_k = t_k t_j$, for $|j - k| > 1$.

In contrast, $\mathscr{H}_n = \langle T_1, \ldots, T_{n-1} \rangle$ with relations $(T_i - q)(T_q + 1) = 0$ together with the braid relations.

Hence, $\mathscr{H}_n \cong R\mathfrak{S}_n$, if $q = 1$.

Motivation

Suppose that $q = p^k$ is a prime power (p prime).
Let $\mathbb{F}_q$ be the finite field with q elements.

Define:

$$G = \mathrm{GL}_n(q) = \text{ invertible } n \times n \text{ matrices}/\mathbb{F}_q$$

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}$$

$$= \text{ upper triangular matrices in } G$$

$$1_B = \text{ trivial representation of } B$$

$$\mathrm{Ind}_B^G(1_B) = \text{ induced representation of } G$$

...an amazing theorem

Let $H_{G,B} = \mathrm{End}_G\big(\mathrm{Ind}_B^G(1_B)\big)$, an R–algebra.

Assume that $R = \mathbb{C}$.

We have the following:

- $\mathbb{C}\mathfrak{S}_n \cong \mathscr{H}_1 \overset{!!}{\cong} \mathscr{H}_q \overset{!}{\cong} H_{G,B}$
- There is a 1-1 correspondence $\chi \leftrightarrow \chi_q$ between the irreducible representations of $\mathfrak{S}_n$ and the irreducible constituents of $\mathrm{Ind}_B^G(1_B)$.
- There is a polynomial $\mathbf{d}_\chi(x)$, for $\chi \in \mathrm{Irr}\,\mathfrak{S}_n$, such that: $\mathbf{d}_\chi(1) = \mathbf{dim}\,\chi$ and $\mathbf{d}_\chi(q) = \mathbf{dim}\,\chi_q$ *for all* $\mathbf{GL}_n(q)$!!!

A basis

- If $w \in \mathfrak{S}_n$ then write $w = t_{i_1} \ldots t_{i_k}$, with k *minimal*.
- Define $T_w = T_{i_1} \ldots T_{i_k}$.
  Then T_w is independent of the choice of $i_1, \ldots, i_k$.
- Further, $\{ T_w : w \in \mathfrak{S}_n \}$ is a basis of $\mathscr{H}$.
- The Hecke algebra $\mathscr{H}_n$ is semisimple iff

$$\prod_{m=1}^{n} (1 + q + \cdots + q^{m-1}) \neq 0.$$

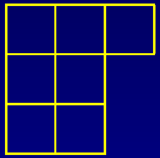
- So, $\mathscr{H}$ need not be semisimple even if $R = \mathbb{C}$!

Representation theory

- Irreducible representations of $\mathfrak{S}_n$ are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$, with $|\lambda| = \sum_i \lambda_i = n$.
- As we might hope, the same is true for $\mathscr{H}$.
- To describe this we need some combinatorics.

Tableaux combinatorics

Fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of n.
We can think of λ as being an array of boxes.
For example, if $\lambda = (3, 2, 2)$ then

$\lambda \leftrightarrow$ 

A λ-tableau is a filling of this diagram with the numbers $1, \ldots, n$. For example,

1	2	3
4	5	
6	7	

1	2	4
3	5	
6	7	

1	2	5
3	4	
6	7	

and

1	4	7
2	5	
3	6	

are all $(3, 2, 2)$-tableau.

...tableaux

The symmetric group $\mathfrak{S}_n$ acts on the set of tableaux by permuting the entries.

Let $\mathfrak{t}^\lambda$ be the λ-tableau with the numbers $1, \dots, n$ entered in order along the rows:

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

If $\mathfrak{t}$ is a λ-tableau we let $d(\mathfrak{t}) \in \mathfrak{S}_n$ be the *unique* permutation such that $\mathfrak{t} = \mathfrak{t}^\lambda \cdot d(\mathfrak{t})$.

Example If $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array}$

then $d(\mathfrak{t}) = (2, 3, 4)(5, 6)$.

Young subgroups

To each partition $\lambda = (\lambda_1, \ldots, \lambda_k$ we can also associate a Young subgroup

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \hookrightarrow \mathfrak{S}_n.$$

Thus, $\mathfrak{S}_\lambda$ is the row stabilizer of $\mathfrak{t}^\lambda$.
To construct representations we need the following element:

$$m_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$$

In fact, $m_\lambda \mathscr{H} = \operatorname{Ind}_{\mathscr{H}_\lambda}^{\mathscr{H}}(1_\lambda)$

A cellular basis

A λ-tableau is standard if its entries increase along the rows and down the columns. For example:

1	2	3
4	5	
6	7	

and

1	3	4
2	6	
5	7	

If $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableaux define

$$m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} m_\lambda T_{d(\mathfrak{t})} \qquad \in \mathscr{H}.$$

Theorem (Murphy)
$\{ m_{\mathfrak{s}\mathfrak{t}} : \mathfrak{s}, \mathfrak{t} \text{ standard } \lambda - \text{tableaux} \}$ is a cellular basis of $\mathscr{H}$.

Cellular algebras (Graham–Lehrer)

In essence, the *cellular* basis $\{m_{\mathfrak{st}}\}$ determines the representation theory of $\mathscr{H}$.

(C1) The map $* : m_{\mathfrak{st}} \mapsto m_{\mathfrak{ts}}$ is an anti–isomorphism.

(C2) Given $\mathfrak{t}$ and $h \in \mathscr{H}$ there exist $r^h_{\mathfrak{tv}} \in R$ such that

$$m_{\mathfrak{st}}h \equiv \sum_{\mathfrak{v}\in \mathrm{Std}(\lambda)} r^h_{\mathfrak{tv}} m_{\mathfrak{sv}} \pmod{\text{higher terms}}$$

Importantly, $r^h_{\mathfrak{tv}}$ is *independent* of $\mathfrak{s}$!

(C1) and (C2) combined give:

(C2)′ $$hm_{\mathfrak{st}} \equiv \sum_{\mathfrak{v}\in \mathrm{Std}(\lambda)} r^h_{\mathfrak{sv}} m_{\mathfrak{vt}} \pmod{\text{higher terms}}$$

Specht modules

The Specht module $S(\lambda)$ is the free R–module with basis $\{\, m_{\mathfrak{t}} : \mathfrak{t} \in \mathrm{Std}(\lambda)\,\}$ and with $\mathscr{H}$-action:

$$m_{\mathfrak{t}} h = \sum_{\mathfrak{v} \in \mathrm{Std}(\lambda)} r_{\mathfrak{t}\mathfrak{v}}^{h} m_{\mathfrak{v}}.$$

Importantly, $S(\lambda)$ has a natural bilinear form $\langle\ ,\ \rangle$. To define $\langle\ ,\ \rangle$ it is enough to specify $\langle m_{\mathfrak{t}}, m_{\mathfrak{u}}\rangle$.

$$m_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{u}\mathfrak{v}} \equiv \langle m_{\mathfrak{t}}, m_{\mathfrak{u}}\rangle m_{\mathfrak{s}\mathfrak{v}} \pmod{\text{higher terms}}$$

This equation defines a bilinear form on $S(\lambda)$.

Simple modules

The bilinear form $\langle\ ,\ \rangle$ is associative in the sense that $\langle xh, y\rangle = \langle x, yh^*\rangle$, for all $x, y \in S(\lambda)$, $h \in \mathscr{H}$.

$\operatorname{rad} S(\lambda) = \{\, x \in S(\lambda) : \langle x, y\rangle = 0 \text{ for all } y \in S(\lambda)\,\}$ is an $\mathscr{H}$–submodule of $S(\lambda)$.

Define $D(\lambda) = S(\lambda)/\operatorname{rad} S(\lambda)$.

Theorem Suppose R is a field. Then $\{\, D(\lambda) : D(\lambda) \neq 0\,\}$ is a complete set of pairwise non–isomorphic irreducible $\mathscr{H}$–modules.

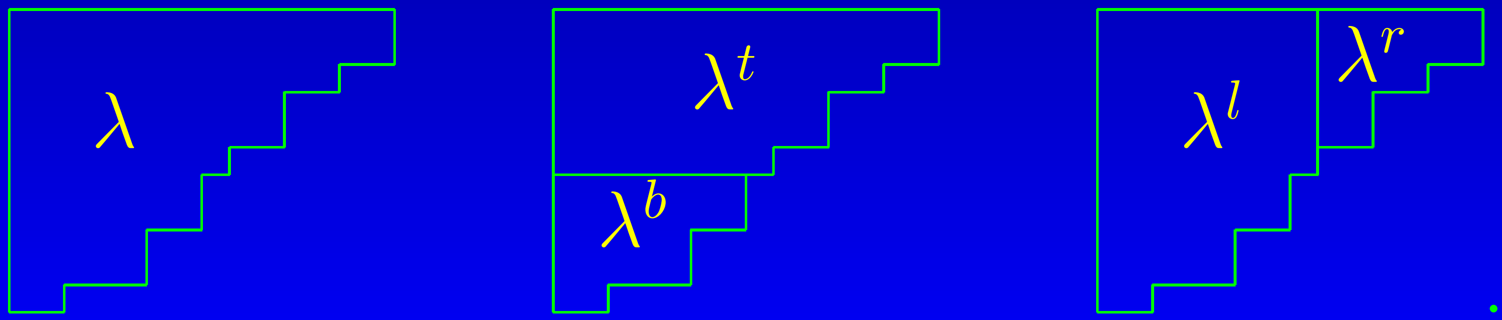
Define $e = \min\{\, k > 0 : 1 + q + \cdots + q^{k-1} = 0\,\}$. Dipper and James showed that $D(\lambda) \neq 0$ if and only if $\lambda_i - \lambda_{i+1} < e$ for all $i \geq 1$.

Specht module homomorphisms

Theorem
Let λ and μ be partitions and assume that $q \neq -1$. Suppose that $\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s$. Then

$$\operatorname{Hom}_{\mathscr{H}_n}\big(S(\mu), S(\lambda)\big) \cong \operatorname{Hom}_{\mathscr{H}_{n-m}}\big(S(\mu^t), S(\lambda^t)\big) \otimes \operatorname{Hom}_{\mathscr{H}_m}\big(S(\mu^b), S(\lambda^b)\big).$$

where $m = |\lambda^b|$.
Pictorially, this can be viewed as follows:



Idea of proof

Lemma Suppose that $q \neq -1$. Then $\operatorname{Hom}_{\mathscr{H}}(M(\mu), S(\lambda))$ is free as an R-module with basis $\{\, \varphi_T : T$ *semistandard* of type $\mu \,\}$.

Basically, $\varphi_T(m_\mu h) = \sum_{\mathfrak{t} \sim_\mu T} m_{\mathfrak{t}} h$.

Example

Let $\lambda = (5, 4, 3, 3, 2)$ and $\mu = (4, 4, 4, 2, 2, 1)$.
The semistandard λ–tableaux of type μ are:

$$
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\\hline 2&2&2&3\\\cline{1-4} 3&3&3\\\cline{1-3} 4&4&5\\\cline{1-3} 5&6\\\cline{1-2}\end{array}
\quad
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\\hline 2&2&2&3\\\cline{1-4} 3&3&3\\\cline{1-3} 4&4&6\\\cline{1-3} 5&5\\\cline{1-2}\end{array}\ ,
\quad
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&3\\\hline 2&2&2&2\\\cline{1-4} 3&3&3\\\cline{1-3} 4&4&5\\\cline{1-3} 5&6\\\cline{1-2}\end{array}\ ,
\quad
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&3\\\hline 2&2&2&2\\\cline{1-4} 3&3&3\\\cline{1-3} 4&4&6\\\cline{1-3} 5&5\\\cline{1-2}\end{array}\ .
$$

We take $s = 3$.

...idea of proof

From the combinatorics and the last lemma we find

$$\mathrm{Hom}_{\mathscr{H}_n}\big(M(\mu), S(\lambda)\big) \cong \mathrm{Hom}_{\mathscr{H}_{n-m}}\big(M(\mu^t), S(\lambda^t)\big) \otimes \mathrm{Hom}_{\mathscr{H}_m}\big(M(\mu^b), S(\lambda^b)\big),$$

via $\varphi_T \mapsto \varphi_{T^t} \otimes \varphi_{T^b}$.
On the other hand, we have a surjection

$$M(\mu) \to S(\mu).$$

It is now *just* a matter of "lifting" and "pushing" Specht module maps through this surjection.

This takes a fair amount of work, but is not so bad...

The proof yields an explicit bijection.

Generalizations — I

Let $\mathscr{S}$ be the q–Schur algebra.

We have exactly the same result but with Specht modules replaced with Weyl modules:

Theorem Suppose that $\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s$. For any R and q we have

$$\begin{aligned}&\operatorname{Hom}_{\mathscr{S}}\big(\Delta(\mu), \Delta(\lambda)\big) \cong \\ &\quad \operatorname{Hom}_{\mathscr{S}}\big(\Delta(\mu^t), \Delta(\lambda^t)\big) \otimes \operatorname{Hom}_{\mathscr{S}}\big(\Delta(\mu^b), \Delta(\lambda^b)\big).\end{aligned}$$

Generalizations — II

Theorem (Donkin) Suppose that R is a field and that $\lambda_1+\cdots+\lambda_s=\mu_1+\cdots+\mu_s$. Then

$$\operatorname{Ext}^k_{\mathscr{S}}\big(\Delta(\mu),\Delta(\lambda)\big)\cong \bigoplus_{i+j=k}\operatorname{Ext}^i_{\mathscr{S}}\big(\Delta(\mu^t),\Delta(\lambda^t)\big)\otimes\operatorname{Ext}^j_{\mathscr{S}}\big(\Delta(\mu^b),\Delta(\lambda^b)\big).$$

Theorem (Donkin)
Suppose that R is a field, $e>2$ and that $\lambda_1+\cdots+\lambda_s=\mu_1+\cdots+\mu_s$. Then

$$\operatorname{Ext}^k_{\mathscr{H}}\big(S(\mu),S(\lambda)\big)\cong \bigoplus_{i+j=k}\operatorname{Ext}^i_{\mathscr{H}}\big(S(\mu^t),S(\lambda^t)\big)\otimes\operatorname{Ext}^j_{\mathscr{H}}\big(S(\mu^b),S(\lambda^b)\big)$$

for $0\le k<e-2$.

Generalizations — III

Theorem (CPS,Donkin) Suppose that R is a field and that $\lambda_1+\cdots+\lambda_s=\mu_1+\cdots+\mu_s$. Then

$$\operatorname{Ext}^k_{\mathscr{S}}\big(L(\mu),\Delta(\lambda)\big)\cong \bigoplus_{i+j=k}\operatorname{Ext}^i_{\mathscr{S}}\big(\Delta(\mu^t),L(\lambda^t)\big)\otimes\operatorname{Ext}^j_{\mathscr{S}}\big(\Delta(\mu^b),L(\lambda^b)\big).$$

Theorem (Donkin, Lyle-M.)
Suppose that R is a field, $e>2$, and that $\lambda_1+\cdots+\lambda_s=\mu_1+\cdots+\mu_s$. Then

$$\operatorname{Ext}^k_{\mathscr{H}}\big(S(\mu),D(\lambda)\big)\cong \bigoplus_{i+j=k}\operatorname{Ext}^i_{\mathscr{H}}\big(S(\mu^t),D(\lambda^t)\big)\otimes\operatorname{Ext}^j_{\mathscr{H}}\big(S(\mu^b),D(\lambda^b)\big)$$

for $0\le k<e-2$.

Generalizations — IV

If V is an $\mathscr{S}$–module let $\operatorname{Ch} V = \sum_\nu (\dim V_\nu) e^\nu$ be its formal character.
So $\operatorname{Ch} \Delta(\lambda)$ is given by the Weyl character formula.

$$\operatorname{Ch} L(\lambda) = \sum_{\substack{\mu \\ k \geq 0}} (-1)^k \dim \operatorname{Ext}^k_{\mathscr{S}} \big(\Delta(\mu), L(\lambda)\big) \operatorname{Ch} \Delta(\mu).$$

Following CPS define $Q_{\mu\lambda}(t) \in \mathbb{N}_0[t]$ by

$$Q_{\mu\lambda}(t) = \sum_{k \geq 0} \dim \operatorname{Ext}^k_{\mathscr{S}} \big(\Delta(\mu), L(\lambda)\big)\, t^k.$$

So $\operatorname{Ch} L(\lambda) = \sum_\mu Q_{\mu\lambda}(-1) \operatorname{Ch} \Delta(\mu)$.

...generalizations — IV

From work of Varagnolo–Vasserot (and Leclerc–Thibon), it follows that $Q_{\mu\lambda}(t)$ is a (renormalized) inverse parabolic Kazhdan–Lusztig polynomial.

Let $d_{\mu\lambda}(t)$ be the corresponding parabolic Kazhdan–Lusztig polynomials.

Then:

- (Varagnolo–Vasserot) $d_{\mu\lambda}(1) = [\Delta(\mu) : L(\lambda)]$.
- (Chuang–Miyachi–Tan)
  Suppose $\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s$. Then

$$d_{\mu\lambda}(t) = d_{\mu^t\lambda^t}(t) d_{\mu^b\lambda^b}(t).$$