Complexity of nilpotent orbits and the Kostant-Sekiguchi correspondence

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Donald R. King
Mathematics Department
Northeastern University
Boston, MA 02115

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Outline of Talk

1. Notation, basic facts and definitions

2. Statement of Main Theorem

3. K-structure of nilpotent orbits (Schmid and Vilonen)

4. Proof of Main Theorem

5. Commutativity of a related Poisson algebra

6. An Application to the G-saturation of the nilradical of a parabolic subalgebra

1. Notation

$\mathfrak{g}$ real simple Lie algebra, Cartan involution θ.

Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

$\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$.

σ is conjugation of $\mathfrak{g}_{\mathbf{C}}$ w.r.t. $\mathfrak{g}$.

$G_{\mathbf{C}}$ is the adjoint group of $\mathfrak{g}_{\mathbf{C}}$.

G, K, $K_{\mathbf{C}}$ conn. subgps corr. to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_{\mathbf{C}}$.

Ω a nilpotent G-orbit in $\mathfrak{g}$.

$\mathcal{O}$ a nilpotent $K_{\mathbf{C}}$-orbit in $\mathfrak{p}_{\mathbf{C}}$.

Kostant-Sekiguchi correspondence

Definition. *(orbit "cores")*

$C(\Omega) :=$
$\{E \in \Omega | \exists$ *sl(2)-triple* $\{H, E, F\} \subset \mathfrak{g},\ \theta(E) = -F\}$.

$C(\mathcal{O}) :=$
$\{e \in \mathcal{O} | \exists$ *normal sl(2)-triple* $\{x, e, f\} \subset \mathfrak{g}_{\mathbf{C}},\ \sigma(e) = f\}$.

$C(\Omega)$ (resp., $C(\mathcal{O})$) is a single K orbit.

The correspondence:

$$\Omega \supset C(\Omega) \ni E \mapsto K_{\mathbf{C}} \cdot \frac{H - i(E+F)}{2} := \mathcal{O}_{\Omega}$$

$(\Omega,\ \mathcal{O}_{\Omega})$ is said to be a KS-pair.

Properties of Kostant-Sekiguchi pairs

Let $(\Omega,\ \mathcal{O})$ be a KS-pair.

(1) $\dim_{\mathbf{R}} \Omega = \dim_{\mathbf{R}} \mathcal{O}$.

(2) Ω and $\mathcal{O}$ lie in the same $G_{\mathbf{C}}$ orbit, $\mathcal{O}_{\mathbf{C}}$.

(3) $\mathcal{O}$ is a Lagrangian submanifold of $\mathcal{O}_{\mathbf{C}}$.

(4) (Vergne) $\exists$ K-equivariant real analytic isomorphism, $\mathcal{V} : \Omega \to \mathcal{O}$.

(5)(Barbasch and Sepanski)

$$\Omega_1 \subset \overline{\Omega_2} \iff \mathcal{O}_1 \subset \overline{\mathcal{O}}_2$$

How does the correspondence affect other invariants of Ω and $\mathcal{O}$ that are significant in representation theory?

The Big Picture

$$\Omega \quad \xrightarrow{\mathcal{V}} \quad \mathcal{O}$$

symplectic manifold
Hamiltonian K-space

quasi-affine variety
algebraic $K_{\mathbf{C}}$-action

Invariants

$\swarrow \quad \searrow$

K-rank

K-corank

$K_{\mathbf{C}}$-rank

$K_{\mathbf{C}}$-complexity

Notions of rank

rank of K-action on Ω, $r_K(\Omega)$:=rank K - rank $K^{E'}$ where $E' \in \Omega$ and the orbit $K \cdot E'$ is generic.

$R[\mathcal{O}]$ denotes the ring of regular functions on $\mathcal{O}$. If $\lambda \in \widehat{K}$, $R[O]_\lambda$ is the λ-isotypic component. $m_{\mathcal{O}}(\lambda)$ is the multiplicity of λ in $R[\mathcal{O}]$.

$\Gamma(\mathcal{O}) := \{\lambda \in \widehat{K} | R[O]_\lambda \neq (0)\}$ is a finitely generated semigroup in the set of dominant weights of $K_{\mathbf{C}}$.

The rank of the $K_{\mathbf{C}}$ action on $\mathcal{O}$ is the rank of $\Gamma(\mathcal{O})$. It is denoted $r_{K_{\mathbf{C}}}(\mathcal{O})$.

The corank of the K action on Ω

Let w_Ω be the canonical symp. form on Ω and $\Phi : \Omega \to \mathfrak{k}^*$ denote the moment map.

Lemma. *Let $E' \in \Omega$. Set $W = T_{E'}(K \cdot E')$ and $W^\perp$ equal to the orthogonal complement (with respect to $w_\Omega|_{E'}$) of W inside $T_{E'}(\Omega)$. Then $T_{E'}(\Omega)$ is a sum of symplectic vector spaces:*

$$\frac{W}{W \cap W^\perp} \bigoplus \Big((W \cap W^\perp) \oplus (W \cap W^\perp)^* \Big) \bigoplus \frac{W^\perp}{W \cap W^\perp}.$$

where,

(1) $\mathfrak{k}^{\Phi(E')}/\mathfrak{k}^{E'} \simeq W \cap W^\perp$ (as $\mathfrak{k}^{E'}$ modules) and

(2) $\mathfrak{k}/\mathfrak{k}^{\Phi(E')} \simeq \frac{W}{W \cap W^\perp}$ (as $\mathfrak{k}^{E'}$ modules).

Proposition. *(Vinberg)*
$\exists$ open dense subset $U \subset \Omega$ such that for all $E' \in U$, the non-negative integer

$$\dim \frac{T_{E'}(K \cdot E')^{\perp}}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^{\perp}},$$

has a constant value. We denote this value by $2\widetilde{\epsilon}_K(\Omega)$. The integer $2\widetilde{\epsilon}_K(\Omega)$ is said to be the corank of the action of K on Ω.

Remark. *If $\widetilde{\epsilon}_K(\Omega) = 0$, Ω is said to be multiplicity free as a Hamiltonian K-space (that is, $T_{E'}(K \cdot E')$ is coisotropic in $T_{E'}(\Omega)$ for a dense open subset of elements of Ω).*

Multiplicity-free Hamiltonian K-spaces

Let $\mathbf{X}$ be a Hamiltonian K-space; $C^\infty(\mathbf{X})^K$ be the Poisson algebra of K-invariant smooth functions on $\mathbf{X}$; and $\mathfrak{A}(\mathbf{X})^K_{ra}$ be the Poisson algebra of K-invariant real analytic functions on $\mathbf{X}$.

Guillemin and Sternberg have shown:

$\mathbf{X}$ is mult. free $\iff$ $C^\infty(\mathbf{X})^K$ is comm.

$\widetilde{\epsilon}_K(\Omega)$ "should" measure the failure of $C^\infty(\Omega)^K$ to be commutative. We can show that $\widetilde{\epsilon}_K(\Omega)$ is a "rough" measure of the failure of $\mathfrak{A}(\Omega)^K_{ra}$ to be commutative.

Definition. *Let $d = d_\Omega$ be the maximum dimension of a K-orbit in Ω and d_Φ denote the maximum dimension of a K-orbit in $\Phi(\Omega)$.*

$$\Omega_d = \{E' \in \Omega | \dim K \cdot E' = d\}$$

$$\Omega_\Phi = \{E' \in \Omega | \dim K \cdot \Phi(E') = d_\Phi\}$$

$$= \{E' \in \Omega | \dim \mathfrak{k}^{\Phi(E')} \text{ is minimum}\}.$$

Remark. *The subset U in the proposition on slide 9 can be chosen so that $U \subset \Omega_d \cap \Omega_\Phi$.*

The complexity of the $K_{\mathbf{C}}$ action on $\mathcal{O}$

Definition. *If $B_{\mathfrak{k}} \subset K_{\mathbf{C}}$ is a Borel subgroup, then the $c_{K_{\mathbf{C}}}(\mathcal{O})$, the complexity of the $K_{\mathbf{C}}$ action, is the codimension of a generic $B_{\mathfrak{k}}$-orbit in $\mathcal{O}$. It is also the transcendence degree over $\mathbf{C}$ of the field of $B_{\mathfrak{k}}$-invariant rational functions on $\mathcal{O}$.*

Proposition. *(Panyushev) $c = c_{K_{\mathbf{C}}}(\mathcal{O})$ is the smallest non-negative integer such that for all $\lambda \in \Gamma(\mathcal{O})$:*

$m_{\mathcal{O}}(n\lambda)$ *grows no faster than* n^c *as* $n \to \infty$

Spherical nilpotent orbits

Definition. *If* $c_{K_{\mathbf{C}}}(\mathcal{O}) = 0$, $\mathcal{O}$ *is said to spherical (for* $K_{\mathbf{C}}$*).*

(McGovern and Panyushev) Classification of spherical orbits for $\mathfrak{g}$ simple and complex.

(King, 2003) Classification of spherical orbits for $\mathfrak{g}$ simple and real.

2. Main Theorem

Theorem. *Let* $(\Omega, \mathcal{O})$ *be a Kostant-Sekiguchi pair, then*

(a) $r_{K_{\mathbf{C}}}(\mathcal{O}) = r_K(\Omega)$;

(b) $c_{K_{\mathbf{C}}}(\mathcal{O}) = \widetilde{\epsilon}_K(\Omega)$.

Corollary. *(King, TAMS (2002))*
Ω *is multiplicity free* $\iff$ $\mathcal{O}$ *is spherical.*

Example. *When* $\mathfrak{g}$ *is simple, the minimal non-zero* Ω *is multiplicity free, and* $\mathcal{O}_\Omega$ *is spherical.*

Motivation

1. The previous corollary.

2. I. V. Mykytyuk's result:

$$\widetilde{\epsilon}_G(T^*(G/K)) = c_{G_{\mathbf{C}}}(G_{\mathbf{C}}/K_{\mathbf{C}}),$$

where K and G are compact groups, $K \subset G$, and $K_{\mathbf{C}}$ and $G_{\mathbf{C}}$ are there complexifications.

Actions of Borel Subgroups on Homogeneous Spaces of Reductive Complex Lie Groups and Integrability, preprint, 2001.

Ingredients in the proof of main theorem

1. The Vergne real analytic isomorphism

2. Results of Schmid and Vilonen on the K structure of $\mathcal{O}$

3. Results of Panyushev on rank and complexity of K_{C} actions

4. Results of A. T. Huckleberry and T. Wurzbacher on K actions on symplectic manifolds

3. The K structure of $\mathcal{O}$

Let $E \in C(\Omega)$, $e \in C(\mathcal{O})$, $\{H,\ E,\ F\} \leftrightarrow \{x,\ e,\ f\}$. Set $\mathfrak{s} := \mathrm{R}H + \mathrm{R}E + \mathrm{R}F$.

Proposition. *(Schmid-Vilonen in "Geometry of nilpotent orbits")*

If $V_{\mathcal{O}}(\mathfrak{s}) = [\mathfrak{k}_{\mathrm{C}},\ e]/[\mathfrak{k},\ e]$, *then* $\exists$ K*-invariant real analytic isomorphism:*

$$\mathcal{O} \simeq K \times_{K^{\mathfrak{s}}} V_{\mathcal{O}}(\mathfrak{s}) \simeq T^*_{C(\mathcal{O})}(\mathcal{O}).$$

In addition, there is an isomorphism of $K^{\mathfrak{s}}$ modules over R:

$$V_{\mathcal{O}}(\mathfrak{s}) \simeq \mathfrak{k}^x/\mathfrak{k}^{\mathfrak{s}} \oplus Z.$$

4. Proof of Main Theorem

The spaces $\mathfrak{k}^x/\mathfrak{k}^{\mathfrak{s}}$ and Z appear in results of Panyushev on the rank and complexity of $\mathcal{O}$.

Thus one can compute $n_{\mathcal{O}} = n_{\mathcal{O},\ K}$, the real codimension of the largest K-orbit in $\mathcal{O}$, in terms of the rank and complexity, i.e., on the one hand,

$$n_{\mathcal{O}} = r_{K_{\mathbf{C}}}(\mathcal{O}) + 2c_{K_{\mathbf{C}}}(\mathcal{O}). \qquad (1)$$

Proof. Combine results of Schmid-Vilonen with those of Panyushev. $\square$

Another formula for $n_{\mathcal{O}}$

Clearly, $n_{\mathcal{O}} = n_{\Omega}$. Recall the vector space dec. of $T_{E'}(\Omega)$ in terms of $W = T_{E'}(K \cdot E')$:

$$\frac{W}{W \cap W^{\perp}} \bigoplus \Big((W \cap W^{\perp}) \oplus (W \cap W^{\perp})^{*} \Big) \bigoplus \frac{W^{\perp}}{W \cap W^{\perp}}$$

where,

(1) $\mathfrak{k}^{\Phi(E')}/\mathfrak{k}^{E'} \simeq W \cap W^{\perp}$ (as $\mathfrak{k}^{E'}$ modules) and

(2) $\mathfrak{k}/\mathfrak{k}^{\Phi(E')} \simeq \frac{W}{W \cap W^{\perp}}$ (as $\mathfrak{k}^{E'}$ modules).

$$\dim \Omega = \dim T_{E'}(\Omega)$$

$$= 2 \dim \left(\mathfrak{k}^{\Phi(E')}/\mathfrak{k}^{E'}\right) + \dim \left(\mathfrak{k}/\mathfrak{k}^{\Phi(E')}\right) + \dim \frac{W^{\perp}}{W \cap W^{\perp}}$$

$$= \dim \mathfrak{k}^{\Phi(E')} - \dim \mathfrak{k}^{E'} + \dim \mathfrak{k} - \dim \mathfrak{k}^{E'} + \dim \tfrac{W^{\perp}}{W \cap W^{\perp}}$$

One shows that for a dense open subset $\widetilde{U} \subset \Omega$, $E' \in \widetilde{U}$ implies:

$$\dim \mathfrak{k}^{\Phi(E')} - \dim \mathfrak{k}^{E'} = \text{rank } \mathfrak{k}^{\Phi(E')} - \text{rank } \mathfrak{k}^{E'} = r_K(\Omega$$

and

$$\dim \frac{W^{\perp}}{W \cap W^{\perp}} = 2\widetilde{\epsilon}_K(\Omega).$$

The preceding facts depend on work of Huckleberry and Wurzbacher.

One conclude that

$$n_{\mathcal{O}} = r_K(\Omega) + 2\tilde{\epsilon}_K(\Omega). \tag{2}$$

Once it is shown that $r_K(\Omega) = r_{K_{\mathbf{C}}}(\mathcal{O})$, it follows from (1) and (2) that $\tilde{\epsilon}_K(\Omega) = c_{K_{\mathbf{C}}}(\mathcal{O})$.

5. Commutativity of $\mathfrak{A}(\Omega)^K_{ra}$

Definition. *Suppose* M *is a smooth manifold and* $p \in \mathrm{M}$*.* $f_1, \ldots, f_n \in C^\infty(\mathrm{M})$ *are* **functionally independent** *at* p *if* $df_1, \ldots, df_n$ *are linearly independent at* p*.*

Let $\mathfrak{Z} = Z\left(\mathfrak{A}(\Omega)^K_{ra}\right)$ denote the center of the Poisson algebra $\mathfrak{A}(\Omega)^K_{ra}$.

Definition. *Let* $\alpha_K(\Omega)$ *be the largest non negative integer* α *such that* $\exists$ *dense open set* $U^\flat \subset \Omega$ *and* $f_1, \ldots, f_\alpha \in \mathfrak{A}(\Omega)^K_{ra}$ *such that:*

(1) $f_1, \ldots, f_\alpha$ *are f.i. at each* p *in* $U^\flat$ *and*

(2) no nontrivial linear combination of the f_i *belongs to* $\mathfrak{Z}$*.*

Remark. $\alpha = \alpha_K(\Omega)$ *is well defined since the intersection of any two dense open sets is open and dense. Clearly* $\alpha \leq \dim \Omega$*.*

$\alpha_K(\Omega)$ is a measure of the size of $\mathfrak{A}(\Omega)^K_{ra}$ relative to its center. In fact, $\alpha_K(\Omega) = 0 \iff \Omega$ *is multiplicity free*.

$\alpha_K(\Omega)$ is related to $\widetilde{\epsilon}_K(\Omega)$ because:

Proposition. *Let* $n = n_{\{\Omega,K\}}$ *be the codimension of a* K*-orbit of maximal dimension in* Ω*.* $\exists$ *open dense subset* $U^\sharp \subset \Omega$ *and* $f_1, \dots, f_n$ *in* $\mathfrak{A}(\Omega)^K_{ra}$ *which are f.i. on* $U^\sharp$*.*

More on the Poisson algebra $C^\infty(\Omega)^K$

Definition. *$f \in C^\infty(\Omega)$ is* **collective** *if $f = g \circ \Phi$ for some $g \in C^\infty(\mathfrak{k}^*)$. $\mathfrak{B}$ denotes the space of collective functions.*

For $f \in C^\infty(\Omega)$, X_f denotes the Hamiltonian vector field associated to f by w_Ω. That is, for all smooth vector fields Y on Ω:

$$df(Y) = w_\Omega(Y,\ X_f).$$

$C^\infty(\Omega)$ is a Poisson algebra under the Poisson bracket $\{\cdot,\ \cdot\}$ defined as follows. For $f,\ g \in C^\infty(\Omega)$,

$$\{f,\ g\} = w_\Omega(X_f,\ X_g).$$

Since $C^\infty(\Omega)^K$ is the centralizer of $\mathfrak{B}$ in $C^\infty(\Omega)$, the proposition on slide 23 implies:

Lemma. *For E' in a certain dense open subset of Ω and $W = T_{E'}(K \cdot E')$:*

(a) W is spanned by: $\{X_f(E') | f \in \mathfrak{B}\}$.

(b) $W^\perp$ is spanned by: $\{X_f(E') | f \in \mathfrak{A}_{ra}^K\}$.

(c) If $M_{E'}$ denotes a complement of $W \cap W^\perp$ in $W^\perp$, then $M_{E'}$ is spanned by:

$$\{X_f(E') | f \in \mathfrak{A}_{ra}^k,\ f \notin \mathfrak{Z}\}.$$

Unfortunately, we only have:

$$\alpha_K(\Omega) \geq 2\tilde{\epsilon}_K(\Omega).$$

However,

Theorem. *If* $s = 2\tilde{\epsilon}_K(\Omega)$, $\exists$ *open dense subset* U^* *of* Ω, *s.t. for some* $f_1, \ldots, f_s \in \mathfrak{A}_{ra}^K$:

(1) $f_1, \ldots, f_s$ *are f.i. on* U^*;

(2) no nontrivial linear combination of the f_i *belongs to* $\mathfrak{Z}$; *and*

(3) $\forall p \in U^*$, $f_1, \ldots, f_s$ *are f.i. of* $\mathfrak{B}$ *at* p.

s *is the largest integer for which* $\exists$ s *functions* $f_1, \ldots, f_s \in \mathfrak{A}_{ra}^K$ *and an open dense subset* U *satisfying (1), (2) and (3).*

Finally, we obtain:

Corollary. *Let* $(\Omega, \mathcal{O})$ *be a KS-pair, then* $2c_{K_{\mathbf{C}}}(\mathcal{O})$ *is the largest integer* n *s.t.* $\exists\ f_1, \ldots, f_n \in \mathfrak{A}_{ra}^K$ *satisfying (1), (2) and (3) of the previous theorem (on slide 26).*

Thus, in a weak sense, the integer $c_{K_{\mathbf{C}}}(\mathcal{O})$ measures the failure of the Poisson algebra of K-invariant real analytic functions on Ω to be commutative.

6. An application to the G-saturation of the nilradical of a par. sub. of $\mathfrak{g}$

Proposition. *Let* $(\Omega,\ \mathcal{O})$ *be a KS-pair. Suppose that* $P = MAN \subset G$ *(with* $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$*) s.t.* Ω *is an open subset of* $G \cdot \mathfrak{n}$*. Then*

(a) $c_{K_{\mathbf{C}}}(\mathcal{O}) = c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$ *and*

(b) $r_{K_{\mathbf{C}}}(\mathcal{O}) = r_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$*.*

Remark. *The integer* $c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$ *can be calculated by a result of Heckman.*

Example. $\mathfrak{g} = sl(n,\ \mathbf{R})$*. For each* Ω *there is a* $\mathfrak{p}$ *such that* Ω *is open in* $G \cdot \mathfrak{n}$*. Therefore, the complexity and rank of the corresponding* $K_{\mathbf{C}}$*-orbit* $\mathcal{O}$ *can be computed.*

Some related open questions

1. For general $\mathcal{O}$, determine $r_{K_{\mathbf{C}}}(\mathcal{O})$ and $c_{K_{\mathbf{C}}}(\mathcal{O})$.

2. Find a set of generators for $\Gamma(\mathcal{O})$. (For $\mathfrak{g}$ simple and complex, these generators have been found by McGovern.) The orbit method suggests that

$$\Gamma(\mathcal{O}) \subset \sqrt{-1}\Phi(\Omega) \cap \text{ positive Weyl chamber}$$

3. Determine $\Gamma(\mathcal{O})$, $\Gamma(\overline{\mathcal{O}})$, and $\Gamma(\overline{\mathcal{O}}^n)$.