Notations:

- G : complex s.s. algebraic group;
- $H \subset G$: a spherical subgroup (see below);
- $B \subset G$: a Borel subgroup s.t. $BH \subset G$ dense;
- $\mathfrak{B}_H \supset \mathfrak{B}_H(1)$: set of B-orbits in G/H (resp. codim. 1-orbits);
- $T \subset B$: a maximal torus which we *will* fix;
- $R \supset R^+$: root system of (G, T) and +-ve system;
- $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$;
- $\alpha^\vee \in X_*(T)$ & $\mathbb{G}_m^\alpha$: coroot corr. to $\alpha \in R$ and its image in T;
- U_α : one-parameter unipotent subgroup corr. to α;
- $W := N_G(T)/T$: the Weyl group of G;
- $P_\alpha \subset G$: min'l parabolic subgroup corr. to $\alpha \in R$.

Distinguished orbits in regular varieties

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The goal of this talk is to generalize the notion of θ-stable maximal torus from symmetric spaces to much wider class of homogeneous spaces.

Generalized θ-stable maximal torus would help us to

1) do harmonic analysis on the coset space;

2) classify some class of homogeneous spaces.

Generalization:

Symmetric spaces: Spherical G/H

$+\ \theta \in \mathrm{Aut}G$ **s.t.** $\theta^2 = \mathrm{id}, H = G^\theta$

$$\Downarrow$$

Regular varieties: Spherical G/H **s.t.**

$H = N_G^\sharp(H)$

$N_G^\sharp(H) := \{g \in N_G(H); g$ **fixes** $\mathfrak{B}_H(1)\} \subset N_G(H)$

Remark: $\mathrm{Aut}_G G/H = N_G(H)/H$

Definition of "θ-stable" torus:

Symmetric spaces: maximal torus T s.t.
$\theta(T) = T$

$$\Downarrow$$

Regular varieties: difficult since there is NO θ!!!

Char'zation of regular varieries (after Luna)

The dense open G-orbit of a G-variety X s.t.

- **Smooth projective with finite number of G-orbits;**
- **Closed G-orbit is unique;**
- **$X \backslash (G/H)$ is a union of G-stable ncd.**

Examples of regular varieties:

1) $\forall$ **adjoint algebraic symmetric spaces;**

2) $(SL_{p+1}(\mathbb{C}) \times SL_p(\mathbb{C}))/\,SL_p(\mathbb{C})$**;**

3) $SL_n(\mathbb{C})/N_G(GL_p(\mathbb{C}) \times SL_{p+q-1}(\mathbb{C}))$ **for** $n = 2p + q$**;**

4) $G = Sp(4)^*(\supset Sp(2) \times Sp(2)) \supset Sp(2) \times O(2) = H$**.**

*measured by size

Main ingredients:

- W-action on $\mathfrak{B}_H$ (due to Knop)
- Gen. max'lly split torus (due to Knop)
- Boundary behaviour of orbits (due to Brion)
- Gen. notion of real roots

Richardson-Springer corr.† (Kostant etc...)

T: θ-stable max'l torus s.t. $\dim T^\theta$: min.

$$\{\theta\text{-stable max'l torus}\}/\mathrm{Ad}(H)$$
$$\Leftrightarrow T\backslash\{g \in G; g\theta(g^{-1}) \in N_G(T)\}/H$$
$$\Leftrightarrow \mathfrak{B}_H (= \{B\text{-orbits in } G/H\})$$

$$gTg^{-1} \leftrightarrow g \bmod (T \times H) \leftrightarrow Bg[H]$$

is the corr. In part., W acts on $\mathfrak{B}_H$.

Remark on R-S corr.

The element g is taken as the "Cayley transform." by the following:

We have $((wg)T(wg)^{-1})^{-\theta} \subset T^{-\theta}$ for $w \in W$. $\forall\theta$-stable torus $\mapsto$ fixed part of an involution in W and vice versa.

Every involution of a Coxeter group is a product of pairwise commutative reflections (Deodhar).

$\Rightarrow$ $g \mapsto$ a seq. of orth. real roots. $\mapsto$ prod. of simple Cayley trans.

Knop's W-action on $\mathfrak{B}_H$ †

For $Y \in \mathfrak{B}_H$, **we define** rk$Y$ = dim $Y/[B,B]$.

$\exists W$**-action** $\star$ **on** $\mathfrak{B}_H$ **s.t.** rk$Y$ = rk$(w \star Y)$

for all $w \in W$.

$\exists$ **analogue of max'lly split** θ**-stable max'l torus** T **s.t.**

1) $B\dot{w}[H] = w \star (B[H])$ **for all**

$$N_G(T) \ni \dot{w} \mapsto w \in W.$$

2) $N_H(T)/T$ **is max'l &** $N_H(T)[H] \subset T[H]$.

Notations (for LST)

- $P := \{g \in G; gB[H] = B[H]\}$;
- $P = LU$: **Levi dec. s.t.** $T \subset L$;
- $W_L := N_L(T)/T$, $P^- \Leftrightarrow P$: **opposite;**
- x_1 : **unique** P^-**-fixed point in** X.

Local Structure Theorem† (LST due to BLV)

$\exists Z \subset X$: L-stable loc. closed sub. s.t.

- $X_0 := P \times^L Z \hookrightarrow X$: affine open emb.;
- $x_1 \in Z \cong \mathbb{A}^r$ as $L/[L,L]$-varieties;
- $Gx_1 \cong G/P^-$.

LST = boundary behaviour of dense open B-orbit. For $\forall Y \in \mathfrak{B}_H$, we define

$$W(Y) := \{w \in W; \dim X - \dim Y = \ell(w), \overline{B\dot{w}^{-1}Y} = X\},$$

where $\ell : W \to \mathbb{Z}$ is the length func. w.r.t. B. $|W(Y)| \geq 2$ in general.

Define $X_0^w := \dot{w}X_0$, $x_w := \dot{w}x_1$, $X_-^w := X_0^w \cap G/H$, and $\mathbb{O}_w := Bx_w \subset G/P^-$ for each $w \in W$.

Define $\widetilde{W}_H := \{w \in W; (w \star B[H]) = B[H]\}$**;** **Define** $U_w := B \cap \dot{w} U \dot{w}^{-1}$ **and** $U^w := B^- \cap \dot{w} U \dot{w}^{-1}$ **for each** $w \in W$**. For each** $Y \in \mathfrak{B}_H$**,** $w \in W(Y)$**, and** $v \in \widetilde{W}_H$**, there exists a** TU_w**-equivariant fibration**

$$Y \cap X_0^w = U_w \times (Y \cap T U^w \dot{w} \dot{v}[H]) \longrightarrow w\mathbb{O}_1 \longrightarrow\!\!\!\!\rightarrow \mathbb{O}_w.$$

$$\mathrm{f}^Y_{(v,w)} := Y \cap T U^w \dot{w} \dot{v}[H] \text{ (fiber over } x_w\text{)}.$$

Projection $(Y \cap T U^w \dot{w} \dot{v}[H]) \longrightarrow\!\!\!\!\rightarrow T \dot{w} \dot{v}[H]$ **is** <u>**Galois**</u>**.**

Elementary modifications

Let $\alpha \in R$**: simple &** $Y \in \mathfrak{B}_H$ **be s.t.**

$$\dim P_\alpha Y > \dim Y.$$

Pair (α, Y) **is called:**

Type T if $P_\alpha Y$ **has** 3 B**-orbits.**

Type N if $P_\alpha Y$ **has** 2 **orbits &**

$$(P_\alpha Y)^{\mathbb{G}_m^\alpha} \subset Y.$$

Type U if $P_\alpha Y \neq Y$ **& not Type T or N.**

We can also distinguish them via the stabilizer group of P_α **at a point of** Y**.**

Weak order graph

A labeled graph corr. to elm. relations among $\mathfrak{B}_H$.

Vertex := $\mathfrak{B}_H$;

Labels := Simple roots of (G, B)

Edges := $Y, Y' \in \mathfrak{B}_H$ are connected by a edge labeled by $\alpha \in R$ if $Y' \subset P_\alpha Y$. The edge is a simple (resp. double) edge if (α, Y) is Type UT (resp. Type N).

Define an order $\preceq$ on $\mathfrak{B}_H$ gen. by $Y \preceq_\alpha Y' \Leftrightarrow$ 1) $\dim Y' = \dim Y + 1$ 2) Y' and Y are connected by an edge of the weak order graph with label α.

Reflections of $\widetilde{W}_H$

T_0 : identity comp. of the stabilizer of T at $[H]$.

Let $s \in \widetilde{W}_H (\subset W)$ be a reflection s.t. $T_0^s \neq T_0$.

(i.e. $s = s_\alpha$ s.t. $\alpha \in R$ & $\mathbb{G}_m^\alpha \subset T_0$)

$\Downarrow$ **"Reading" from weak order graph**

$\dot{s}_\alpha[H] \subset U_\alpha[H]$.

(Gen.) real roots

$\alpha \in R$ **is called**

psuedo-real root (R_{ps}) **if** $\mathbb{G}_m^\alpha \subset T_0$;

real root (R_{re}) **if** $s_\alpha \in \widetilde{W}_H$;

spherical real root (R_{re}^H)

if $\alpha \in R_{re}$ **&** $\dot{s}_\alpha \in H$;

perpendicular real roots $(R_{re}^\perp)$

if $\alpha \in R_{re}$ **&** $\dot{s}_\alpha \notin H$;

quasi-real root (R_{qu}) **if** $\alpha \in \mathbb{Q}R_{re} \cap R$.

$$R_{ps}{}^* \supset R_{qu} \supset R_{re} = R_{re}^H \oplus R_{re}^\perp$$

*I have no example that $R_{ps} \neq R_{qu}$

Structural result: $\gamma, \delta \in R$ **are called str. orthogonal if** $\gamma \perp \delta$ **&** $\gamma \pm \delta \notin R$.

Lemma

- $\alpha \in R_{re}^{\perp}$ **is str. orth. to** $\forall \beta \in R_{re} \setminus \{\pm\alpha\}$;
- $\widetilde{W}_H$ **preserves both** R_{re}^{H} **and** $R_{re}^{\perp}$;
- $\alpha \in R_{ps} \setminus R_{qu}$ **is str. orth. to** $\forall \beta \in R_{qu}$.

Main Result-I

1) For each $Y \in \mathfrak{B}_H$, $(w, v) \in W(Y) \times \widetilde{W}_H$, $Y \cap \overline{\mathfrak{f}^Y_{(v,w)}}$ contains a unique min'l dim. T-orbit $\mathfrak{o}^Y_v \subset Y$;

2) $\mathfrak{o}^Y_v$ is independent of the choice of $w \in W(Y)$;

3) The set $\mathfrak{T}_H$ consisting of all $\mathfrak{o}^Y_v$ is stable under the action of $N_G(T)$;

4) $\exists$ W-equivariant surjection $\mathfrak{t} : \mathfrak{T}_H \twoheadrightarrow \mathfrak{B}_H$;

Main Result-II

5) $\exists$ **direct factor** $W_H^\perp \subset \widetilde{W}_H$ **s.t.** $W_H^\perp \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus n}$ **and each** $\mathfrak{t}^{-1}(Y)$ **admits a transitive** $W_H^\perp$**-action;**

We call elements of $\mathfrak{T}_H$ **distinguished orbits. For each distinguished** $Tg[H]$, gTg^{-1} **is an analogue of (a** H**-conj. class of)** θ**-stable maximal torus.**

Remarks

1) H-conj. class of "θ-stable torus" corr. to a B-orbit is NOT unique in general (unique in symmetric case by Richardson-Springer).

2) $\mathfrak{t}$ is an isomorphism in symmetric case.

3) $W_H^{\perp}$ is generated by reflections corr. to $R_{re}^{\perp}$.

4) $(T \cap H)$ fixes $w^{-1}\mathfrak{o}$ for each $\mathfrak{o} \in \mathfrak{T}_H$ and $w \in W(\mathfrak{t}(\mathfrak{o}))$.

Explicit relation between $\mathfrak{f}$ and $\mathfrak{o}$

For each $Y \in \mathfrak{B}_H$, we fix $w \in W(Y)$.

T_Y := **identity comp. of stab. of T on $\mathfrak{o}_v^Y$ ($v \in \widetilde{W}_H$).**

$R_Y^w := \{\alpha \in R; \alpha(wT_0w^{-1}) = 1 \& \mathbb{G}_m^\alpha \subset T_Y\}$.

$G_Y^w \supset T$ **: the group corr. to R_Y^w.**

Proposition We have

$$\mathfrak{f}_{(v,w)}^Y \subset G_Y^w \mathfrak{o}_v^Y$$

for every $v \in \widetilde{W}_H$. If G : simply-laced $\Rightarrow$ $\mathfrak{f}_{(v,w)}^Y$ is the open T-orbit in $(G_Y^w \cap B)\mathfrak{o}_v^Y$.

How to prove Main Theorem-I

Lemma **Let** $Y \in \mathfrak{B}_H$**. Let** $w \in W(Y)$**. Let** α **be s.t.** (α, Y) **is of Type U, T, or N. Then, it is of Type T or N**

$\Leftrightarrow$

$ws_\alpha w^{-1} \in \widetilde{W}_{G/H}$ **and** $\alpha(T_Y) = 1$ **holds.**

In conjunction of this lemma, we prove 1) $\mathrm{o}_v^Y \subset \overline{\mathrm{f}_{(v,w)}^Y}$**, 2)** $\mathrm{o}_v^Y \subset G_Y^w \dot{w}\dot{v}[H]$**, and 3)** $\dot{s}\mathrm{f}_{(v,w)}^Y = \mathrm{f}_{(v,s_\alpha w)}^{s_\alpha \star Y}$ **if** (α, Y) **is of Type U.**

How to prove four assertions-Method

Induction on the weak order.

Difficult part is $\Rightarrow$ in Lemma and 3).

Let $Y \in \mathfrak{B}_H$ and $w \in W(Y)$ be s.t. (α, Y) is of Type UTN. Let $Y_+ \subset P_\alpha Y$ be open B-orbit. Then

$$\mathrm{f}^{Y}_{(v,w)} \subset U_\alpha \mathrm{f}^{Y_+}_{(v,s_\alpha w)}$$

holds. (Here we used the assumption that G/H is regular var.)

How to prove four assertions-Observation

Use the component group of $(T \cap H)$, which determines

$$T_0 \prod_{\alpha \in R_{re}^H} \mathbb{G}_m^{\alpha} \subset T.$$

uniquely.

$$\mathrm{f}_{(v,s_\alpha w)}^{Y_+} \subset (G_{Y_+}^{s_\alpha w} \cap B^-) s_\alpha \dot{w} v[H]$$

$$\Downarrow$$

$\dot{w}^{-1}\mathrm{f}_{(v,s_\alpha w)}^{Y_+}$ **admits an action of** $(T \cap H)/T_0$.

How to prove four assertions-Idea

Prove explicit relation between $\mathfrak{f}$ and $\mathbf{o}$

$$\mathfrak{f}^{Y_+}_{(v,s_\alpha w)} \subset (G^{s_\alpha w}_{Y_+} \cap B)\mathbf{o}^{Y_+}_v$$

and compare the min'l dim'l T-orbits and its stabilizer to characterize the position of $\mathfrak{f}^{Y}_{(v,w)}$ via Observation.

Ex. **If $\langle \alpha, \beta \rangle = 1$ for some $\alpha \in R^H_{re}$, then $\alpha^\vee(-1) \subset (T \cap H)/T_0$ acts on $(U_\alpha \backslash 1)T[H]$ freely!**

How to prove Main Theorem-II

Proposition

For each $Y \in \mathfrak{B}_H$, **we define** $R_Y := \{\alpha \in R; \mathbb{G}_m^\alpha \subset T_Y\}$. **Let** $G_Y \supset T$ **be the group corr. to** R_Y.

Then, o_v^Y **is the** T_Y**-fixed point set of**

$$G_Y \mathrm{o}_v^Y \cap Y = G_Y \mathrm{f}_{(v,w)}^Y \cap Y = G_Y \dot{w}\dot{v}[H] \cap Y.$$

Moreover,

$$X \cap \overline{G_Y \mathrm{f}_{(v,w)}^Y} = W_{G_Y} p_{wv}$$

holds. Here W_{G_Y} **is the Weyl group of** G_Y.

How to prove Main Theorem-III

Final Step

We prove that $\mathfrak{T}_H$ is $N(T)$-invariant by

1) Choose nice $w \in W(Y)$ s.t. $\ell(s_\alpha w) < \ell(w)$.

2) Using induction on $\preceq$ according to its type.

By virtue of 1), our construction of $N(T)$-action is compatible with the construction of Knop's W-action.

Further application

We can "generalize" Vogan's c-invariant of symmetric Harish-Chandra modules to general $(\mathfrak{g}, H)$-modules by using the boundary behaviour along the boundary.