Distinguished orbits in regular varieties

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The goal of this talk is to generalize the notion of θ-stable maximal torus from symmetric spaces to much wider class of homogeneous spaces.

Generalized θ-stable maximal torus would help us to

1) do harmonic analysis on the coset space;

2) classify some class of homogeneous spaces.

Generalization:

Symmetric spaces: Spherical G/H

$+$ $\theta \in \mathrm{Aut}G$ **s.t.** $\theta^2 = \mathrm{id}, H = G^\theta$

$$\Downarrow$$

Regular varieties: Spherical G/H **s.t.**

$H = N_G^\natural(H)$

$N_G^\natural(H) := \{g \in N_G(H); g$ **fixes** $\mathfrak{B}_H(1)\} \subset N_G(H)$

Remark: $\mathrm{Aut}_G G/H = N_G(H)/H$

Definition of "θ-stable" torus:

Symmetric spaces: maximal torus T s.t.

$\theta(T) = T$

$$\Downarrow$$

Regular varieties: difficult since there is NO θ!!!

Char'zation of regular varieries (after Luna)

The dense open G-orbit of a G-variety X s.t.

- **Smooth projective with finite number of G-orbits;**
- **Closed G-orbit is unique;**
- **$X\backslash(G/H)$ is a union of G-stable ncd.**

Examples of regular varieties:

1) $\forall$ **adjoint algebraic symmetric spaces;**

2) $(SL_{p+1}(\mathbb{C}) \times SL_p(\mathbb{C}))/\, SL_p(\mathbb{C})$**;**

3) $SL_n(\mathbb{C})/N_G(GL_p(\mathbb{C}) \times SL_{p+q-1}(\mathbb{C}))$ **for** $n = 2p + q$**;**

4) $G = Sp(4)^*(\supset Sp(2) \times Sp(2)) \supset Sp(2) \times O(2) = H$**.**

*measured by size

Main ingredients:

- W-action on $\mathfrak{B}_H$ (due to Knop)
- Gen. max'lly split torus (due to Knop)
- Boundary behaviour of orbits (due to Brion)
- Gen. notion of real roots

Richardson-Springer corr.† (Kostant etc...)

T: θ-stable max'l torus s.t. $\dim T^\theta$: min.

$$\{\theta\text{-stable max'l torus}\}/\mathrm{Ad}(H)$$
$$\Leftrightarrow T\backslash\{g \in G; g\theta(g^{-1}) \in N_G(T)\}/H$$
$$\Leftrightarrow \mathfrak{B}_H (= \{B\text{-orbits in } G/H\})$$

$$gTg^{-1} \leftrightarrow g \bmod (T \times H) \leftrightarrow Bg[H]$$

is the corr. In part., W acts on $\mathfrak{B}_H$.

Remark on R-S corr.

The element g is taken as the "Cayley transform." by the following:

We have $((wg)T(wg)^{-1})^{-\theta} \subset T^{-\theta}$ for $w \in W$. $\forall\theta$-stable torus $\mapsto$ fixed part of an involution in W and vice versa.

Every involution of a Coxeter group is a product of pairwise commutative reflections (Deodhar).

$\Rightarrow$ $g \mapsto$ a seq. of orth. real roots. $\mapsto$ prod. of simple Cayley trans.

Knop's W-action on $\mathfrak{B}_H{}^\dagger$

For $Y \in \mathfrak{B}_H$, we define $\mathrm{rk} Y = \dim Y/[B,B]$.

$\exists W$-action $\star$ on $\mathfrak{B}_H$ s.t. $\mathrm{rk} Y = \mathrm{rk}(w \star Y)$ **for all $w \in W$.**

$\exists$ analogue of max'lly split θ-stable max'l torus T s.t.

1) $B\dot{w}[H] = w \star (B[H])$ **for all**

$$N_G(T) \ni \dot{w} \mapsto w \in W.$$

2) $N_H(T)/T$ **is max'l &** $N_H(T)[H] \subset T[H]$.

Notations (for LST)

- $P := \{g \in G; gB[H] = B[H]\}$;
- $P = LU$: **Levi dec. s.t.** $T \subset L$;
- $W_L := N_L(T)/T$, $P^- \Leftrightarrow P$: **opposite;**
- x_1 : **unique** P^-**-fixed point in** X.

Local Structure Theorem† (LST due to BLV)

$\exists Z \subset X$: L-stable loc. closed sub. s.t.

- $X_0 := P \times^L Z \hookrightarrow X$: affine open emb.;
- $x_1 \in Z \cong \mathbb{A}^r$ as $L/[L,L]$-varieties;
- $Gx_1 \cong G/P^-$.

LST = boundary behaviour of dense open B-orbit. For $\forall Y \in \mathfrak{B}_H$, we define

$$W(Y) := \{w \in W; \dim X - \dim Y = \ell(w), \overline{B\dot{w}^{-1}Y} = X\},$$

where $\ell : W \to \mathbb{Z}$ is the length func. w.r.t. B. $|W(Y)| \geq 2$ in general.

Define $X_0^w := \dot{w}X_0$, $x_w := \dot{w}x_1$, $X_-^w := X_0^w \cap G/H$, and $\mathbb{O}_w := Bx_w \subset G/P^-$ for each $w \in W$.

Define $\widetilde{W}_H := \{w \in W; (w \star B[H]) = B[H]\}$**;**

Define $U_w := B \cap \dot{w} U \dot{w}^{-1}$ **and** $U^w := B^- \cap \dot{w} U \dot{w}^{-1}$ **for each** $w \in W$**. For each** $Y \in \mathfrak{B}_H$**,** $w \in W(Y)$**, and** $v \in \widetilde{W}_H$**, there exists a** TU_w**-equivariant fibration**

$$Y \cap X_0^w = U_w \times (Y \cap TU^w \dot{w}\dot{v}[H]) \longrightarrow w\mathbb{O}_1 \twoheadrightarrow \mathbb{O}_w.$$

$$\mathrm{f}^Y_{(v,w)} := Y \cap TU^w \dot{w}\dot{v}[H] \text{ (fiber over } x_w\text{)}.$$

Projection $(Y \cap TU^w \dot{w}\dot{v}[H]) \twoheadrightarrow T\dot{w}\dot{v}[H]$ **is** <u>**Galois**</u>**.**

Elementary modifications

Let $\alpha \in R$: simple & $Y \in \mathfrak{B}_H$ be s.t.

$$\dim P_\alpha Y > \dim Y .$$

Pair (α, Y) is called:

Type T if $P_\alpha Y$ has 3 B-orbits.

Type N if $P_\alpha Y$ has 2 orbits &

$$(P_\alpha Y)^{\mathbb{G}_m^\alpha} \subset Y .$$

Type U if $P_\alpha Y \neq Y$ & not Type T or N.

We can also distinguish them via the stabilizer group of P_α at a point of Y.

Weak order graph

A labeled graph corr. to elm. relations among $\mathfrak{B}_H$.

Vertex := $\mathfrak{B}_H$;

Labels := Simple roots of (G, B)

Edges := $Y, Y' \in \mathfrak{B}_H$ are connected by a edge labeled by $\alpha \in R$ if $Y' \subset P_\alpha Y$. The edge is a simple (resp. double) edge if (α, Y) is Type UT (resp. Type N).

Define an order $\preceq$ on $\mathfrak{B}_H$ gen. by $Y \preceq_\alpha Y' \Leftrightarrow$ 1) $\dim Y' = \dim Y + 1$ 2) Y' and Y are connected by an edge of the weak order graph with label α.

Reflections of $\widetilde{W}_H$

T_0 : **identity comp. of the stabilizer of** T **at** $[H]$.

Let $s \in \widetilde{W}_H (\subset W)$ **be a reflection s.t.** $T_0^s \neq T_0$.

(i.e. $s = s_\alpha$ **s.t.** $\alpha \in R$ **&** $\mathbb{G}_m^\alpha \subset T_0$**)**

$\Downarrow$ **"Reading" from weak order graph**

$\dot{s}_\alpha[H] \subset U_\alpha[H]$.

(Gen.) real roots

$\alpha \in R$ **is called**

psuedo-real root (R_{ps}) **if** $\mathbb{G}_m^\alpha \subset T_0$;

real root (R_{re}) **if** $s_\alpha \in \widetilde{W}_H$;

spherical real root (R_{re}^H)

if $\alpha \in R_{re}$ **&** $\dot{s}_\alpha \in H$;

perpendicular real roots ($R_{re}^\perp$)

if $\alpha \in R_{re}$ **&** $\dot{s}_\alpha \notin H$;

quasi-real root (R_{qu}) **if** $\alpha \in \mathbb{Q}R_{re} \cap R$.

$$R_{ps}{}^* \supset R_{qu} \supset R_{re} = R_{re}^H \oplus R_{re}^\perp$$

*I have no example that $R_{ps} \neq R_{qu}$

Structural result: $\gamma, \delta \in R$ **are called str. orthogonal if** $\gamma \perp \delta$ **&** $\gamma \pm \delta \notin R$**.**

Lemma

- $\alpha \in R_{re}^{\perp}$ **is str. orth. to** $\forall \beta \in R_{re} \backslash \{\pm\alpha\}$**;**
- $\widetilde{W}_H$ **preserves both** R_{re}^{H} **and** $R_{re}^{\perp}$**;**
- $\alpha \in R_{ps} \backslash R_{qu}$ **is str. orth. to** $\forall \beta \in R_{qu}$**.**

Main Result-I

1) For each $Y \in \mathfrak{B}_H$, $(w, v) \in W(Y) \times \widetilde{W}_H$, $Y \cap \overline{\mathrm{f}^Y_{(v,w)}}$ **contains a unique min'l dim.** T**-orbit** $\mathrm{o}^Y_v \subset Y$;

2) o^Y_v **is independent of the choice of** $w \in W(Y)$;

3) The set $\mathfrak{T}_H$ **consisting of all** o^Y_v **is stable under the action of** $N_G(T)$;

4) $\exists$ W**-equivariant surjection** $\mathrm{t} : \mathfrak{T}_H \twoheadrightarrow \mathfrak{B}_H$;

Main Result-II

5) $\exists$ **direct factor** $W_H^{\perp} \subset \widetilde{W}_H$ **s.t.** $W_H^{\perp} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus n}$ **and each** $\mathfrak{t}^{-1}(Y)$ **admits a transitive** $W_H^{\perp}$**-action;**

We call elements of $\mathfrak{T}_H$ **distinguished orbits. For each distinguished** $Tg[H]$**,** gTg^{-1} **is an analogue of (a** H**-conj. class of)** θ**-stable maximal torus.**

Remarks

1) H-conj. class of "θ-stable torus" corr. to a B-orbit is NOT unique in general (unique in symmetric case by Richardson-Springer).

2) $\mathfrak{t}$ is an isomorphism in symmetric case.

3) $W_H^{\perp}$ is generated by reflections corr. to $R_{re}^{\perp}$.

4) $(T \cap H)$ fixes $w^{-1}\mathfrak{o}$ for each $\mathfrak{o} \in \mathfrak{T}_H$ and $w \in W(\mathfrak{t}(\mathfrak{o}))$.

Explicit relation between f and o

For each $Y \in \mathfrak{B}_H$, we fix $w \in W(Y)$.

$T_Y :=$ identity comp. of stab. of T on o_v^Y ($v \in \widetilde{W}_H$).

$R_Y^w := \{\alpha \in R; \alpha(wT_0w^{-1}) = 1 \& \mathbb{G}_m^\alpha \subset T_Y\}$.

$G_Y^w \supset T$: the group corr. to R_Y^w.

Proposition **We have**

$$\mathrm{f}_{(v,w)}^Y \subset G_Y^w \mathrm{o}_v^Y$$

for every $v \in \widetilde{W}_H$. If G : simply-laced $\Rightarrow$ $\mathrm{f}_{(v,w)}^Y$ is the open T-orbit in $(G_Y^w \cap B)\mathrm{o}_v^Y$.

How to prove Main Theorem-I

Lemma Let $Y \in \mathfrak{B}_H$. Let $w \in W(Y)$. Let α be s.t. (α, Y) is of Type U, T, or N. Then, it is of Type T or N

$\Leftrightarrow$

$ws_\alpha w^{-1} \in \widetilde{W}_{G/H}$ and $\alpha(T_Y) = 1$ holds.

In conjunction of this lemma, we prove

1) $\mathrm{o}_v^Y \subset \overline{\mathrm{f}_{(v,w)}^Y}$, 2) $\mathrm{o}_v^Y \subset G_Y^w \dot{w}\dot{v}[H]$, and

3) $\dot{s}\mathrm{f}_{(v,w)}^Y = \mathrm{f}_{(v,s_\alpha w)}^{s_\alpha \star Y}$ if (α, Y) is of Type U.

How to prove four assertions-Method

Induction on the weak order.

Difficult part is $\Rightarrow$ in Lemma and 3).

Let $Y \in \mathfrak{B}_H$ and $w \in W(Y)$ be s.t. (α, Y) is of Type UTN. Let $Y_+ \subset P_\alpha Y$ be open B-orbit. Then

$$\mathrm{f}_{(v,w)}^{Y} \subset U_\alpha \mathrm{f}_{(v,s_\alpha w)}^{Y_+}$$

holds. (Here we used the assumption that G/H is regular var.)

How to prove four assertions-Observation

Use the component group of $(T \cap H)$**, which determines**

$$T_0 \prod_{\alpha \in R_{re}^H} \mathbb{G}_m^{\alpha} \subset T.$$

uniquely.

$$\mathrm{f}_{(v,s_\alpha w)}^{Y_+} \subset (G_{Y_+}^{s_\alpha w} \cap B^-) s_\alpha \dot{w} v[H]$$

$$\Downarrow$$

$\dot{w}^{-1}\mathrm{f}_{(v,s_\alpha w)}^{Y_+}$ **admits an action of** $(T \cap H)/T_0$**.**

How to prove four assertions-Idea

Prove explicit relation between f and o

$$\mathrm{f}_{(v,s_\alpha w)}^{Y_+} \subset (G_{Y_+}^{s_\alpha w} \cap B)\mathrm{o}_v^{Y_+}$$

and compare the min'l dim'l T-orbits and its stabilizer to characterize the position of $\mathrm{f}_{(v,w)}^{Y}$ via Observation.

Ex. If $\langle \alpha, \beta \rangle = 1$ for some $\alpha \in R_{re}^H$, then $\alpha^\vee(-1) \subset (T \cap H)/T_0$ acts on $(U_\alpha \backslash 1)T[H]$ freely!

How to prove Main Theorem-II

Proposition

For each $Y \in \mathfrak{B}_H$, we define $R_Y := \{\alpha \in R; \mathbb{G}_m^\alpha \subset T_Y\}$. Let $G_Y \supset T$ be the group corr. to R_Y.

Then, o_v^Y is the T_Y-fixed point set of

$$G_Y \mathrm{o}_v^Y \cap Y = G_Y \mathrm{f}_{(v,w)}^Y \cap Y = G_Y \dot{w}\dot{v}[H] \cap Y.$$

Moreover,

$$X \cap \overline{G_Y \mathrm{f}_{(v,w)}^Y} = W_{G_Y} p_{wv}$$

holds. Here W_{G_Y} is the Weyl group of G_Y.

How to prove Main Theorem-III

Final Step

We prove that $\mathfrak{T}_H$ is $N(T)$-invariant by

1) Choose nice $w \in W(Y)$ s.t. $\ell(s_\alpha w) < \ell(w)$.

2) Using induction on $\preceq$ according to its type.

By virtue of 1), our construction of $N(T)$-action is compatible with the construction of Knop's W-action.

Further application

We can "generalize" Vogan's c-invariant of symmetric Harish-Chandra modules to general $(\mathfrak{g}, H)$-modules by using the boundary behaviour along the boundary.