# On the Geometry of Unimodular Equivalence Classes of Bilinear Forms

D.Ž. Đoković
J. Sekiguchi
K. Zhao

$K$ an alg. closed field of characteristic 0

$V$ fin. dim. $K$-vector space
dim$(V) = n = 2m$ or $2m + 1$

$\mathcal{B}$ the space of bilinear forms
$f : V \times V \to K$

$\mathcal{B} = \mathcal{B}^+ \oplus \mathcal{B}^-, \; f = f^+ + f^-$

$\mathcal{B}^+$ symmetric forms

$\mathcal{B}^-$ skew-symmetric forms

$GL_n = GL(V)$

$SL_n = SL(V)$

$\{v_1, v_2, \ldots, v_n\}$ a basis of $V$

$f \in \mathcal{B}$

$A_f = \mathsf{Mat}(f) = (a_{ij})$

$a_{ij} = f(v_i, v_j)$

$\mathcal{B}$ is a $GL_n$-module

$(a, f) \to a \cdot f,\ \ a \in GL_n,\ \ f \in \mathcal{B}$

$(a \cdot f)(x, y) = f(a^{-1}(x), a^{-1}(y)),\ \forall\ x, y \in V$

$\mathcal{B}^+, \mathcal{B}^-$ are submodules

$GL_n$-orbits in $\mathcal{B}$ are equivalence classes of forms

$SL_n \cdot f = \{a \cdot f : a \in SL_n\}$

unimodular or $SL_n$-orbits

$K[\mathcal{B}]$ the $K$-algebra of regular (polynomial) functions on $\mathcal{B}$

$K[\mathcal{B}]^{SL_n}$ the subalgebra of $SL_n$-invariant regular functions

$P(f,t) := \mathrm{det}(A_{f+} - tA_{f-})$

$f \in \mathcal{B}$, $t$ an indeterminate

$P(f,t) = P_0(f) + P_1(f)t^2 + \cdots + P_m(f)t^{2m}$

If $n = 2m$ is even then $P_m(f) = \mathrm{Pf}(A_{f-})^2$.

Pf is the Pfaffian

**Theorem** (Adamovich & Golovina, 1977)

$K[\mathcal{B}]^{SL_n}$ is generated by

$$\begin{array}{ll} P_0, P_1, \ldots, P_m & \text{if } n = 2m+1 \\ P_0, P_1, \ldots, P_{m-1}, \mathrm{Pf} & \text{if } n = 2m. \end{array}$$

In both cases these generators are algebraically independent.

---

The embedding

$$K[\mathcal{B}] \longleftarrow K[\mathcal{B}]^{SL_n}$$

correpoends to the projection map

$$\pi : \mathcal{B} \longrightarrow \mathcal{B}/\!/SL_n \cong K^{m+1}.$$

$\mathcal{B}/\!/SL_n$ is the categorical quotient.

The morphism $\pi$ is onto.

$$\pi(f) = (P_0(f), P_1(f), \ldots, P_m(f)), \quad n = 2m+1$$
$$= (P_0(f), P_1(f), \ldots, \mathrm{Pf}(f^-)), \quad n = 2m$$

The fibers $\pi^{-1}(\pi(f)) \subset \mathcal{B}$ are closed subvarieties of $\mathcal{B}$.

$\mathcal{N} := \pi^{-1}(\pi(0))$ is the **null-cone**.

The fibers are equi-dimensional varieties and all of them have dimension

$$n^2 - m - 1.$$

If $f \in \mathcal{N}$, $f$ is a **null-form**.

**Example.** $n = 2$

$$A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$P(f,t) = \det\left(\begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} - t\begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix}\right)$$

$$P_0(f) = ad - \tfrac{1}{4}(b+c)^2$$

$$\mathrm{Pf}(f^-) = \tfrac{1}{2}(b-c)$$

$\mathcal{N}$ is defined by the equations:

$$b = c, \quad ad = b^2.$$

$\mathcal{N}$ is the union of two $SL_2$-orbits:

1) $\{0\}$

and

2) symmetric matrices of rank 1.

Questions about the $SL_n$-action on $\mathcal{B}$:

1) Is $\mathcal{N}$ the union of only finitely many $SL_n$-orbits?

2) Characterize the closed $SL_n$-orbits.

3) Given $f \in \mathcal{B}$, determine the unique closed $SL_n$-orbit contained in $\pi^{-1}(\pi(f))$.

4) For which $f \in \mathcal{B}$ one has $\pi^{-1}(\pi(f)) = SL_n \cdot f$?

5) Is there an $SL_n$-orbit which is open in $\mathcal{N}$?

6) Is $\mathcal{N}$ irreducible?

7) Construct a Weierstrass section.

**Proposition 2.3**

If $n \geq 3$, $\mathcal{N}$ contains infinitely many $SL_n$-orbits.

**Pf.** $\alpha \in K,\ \alpha^2 \neq -1,\ f_\alpha \in \mathcal{B}$

$$\text{Mat}(f_\alpha) = \begin{pmatrix} A_\alpha & 0 \\ 0 & 0 \end{pmatrix},\ A_\alpha = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}$$

$f_\alpha \cong f_\beta \Rightarrow X A_\alpha X' = A_\beta$ for some $X \in GL_2$

$$A_{\alpha}^{-} = A_{\beta}^{-} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow X \in SL_2$$

$$\beta^2 + 1 = |A_\beta| = |A_\alpha| = \alpha^2 + 1 \Rightarrow \alpha^2 = \beta^2$$

## Theorem 7.3

The orbit $SL_n \cdot f$, $f \neq 0$, is closed in $\mathcal{B}$ iff, with respect to a suitable basis, the matrix of $f$ is the direct sum of

(1) 1-by-1 blocks $(\lambda)$, $\lambda \neq 0$

and

(2) 2-by-2 blocks $\begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}$, $\mu \neq \nu$.

$D(f) :=$ discriminant of

$$|tA_{f+} - A_{f-}|$$

(a polynomial in $t$ of degree $n$)

$\mathcal{B}_{\text{gen}} := \{f \in \mathcal{B} : |A_{f+}| \neq 0 \neq D(f)\}$

**Theorem 7.5**

$\pi^{-1}(\pi(f)) = SL_n \cdot f \Longleftrightarrow f \in \mathcal{B}_{\text{gen}}$

**Proof of** $\Leftarrow$

May assume $A_{f+} = \lambda I_n,\ \lambda \in K^*$.

Define $u^- \in \mathrm{End}(V)$ by

$$f^-(x,y) = f^+(u^-(x),y),\ \forall\ x,y \in V.$$

$\mathrm{Mat}(u^-) = -\lambda^{-1} A_{f-}$

$D(f) \neq 0 \Rightarrow u^-$ has $n$ distinct eigenvalues

$u^- \in \mathfrak{so}_n := \mathfrak{so}(f^+)$ regular ss element

May assume

$$A_{f-} = \bigoplus_{i=1}^{m} \begin{bmatrix} 0 & \mu_i \\ -\mu_i & 0 \end{bmatrix}$$

$$\oplus [0] \text{ if } n = 2m+1$$

$$\mu_i^2 \text{ nonzero \& distinct}$$

Hence

$$(SO_n)_{f^-} = T = \text{a max. torus of } SO_n$$

and

$$(SL_n)_f = (SO_n)_{f^-} = T.$$

Observe

$$\dim(SL_n \cdot f) = n^2 - 1 - m = \dim \pi^{-1}(\pi(f)).$$

How to find the unique closed orbit, say $SL_n \cdot f_c$, contained in the given fiber $\pi^{-1}(\pi(f))$?

If $f \in \mathcal{N}$, then $f_c = 0$. Assume $f \notin \mathcal{N}$.

Find a decomposition

$$V = V_1 \perp V_2 \perp \cdots \perp V_k$$
$$f = f_1 \perp f_2 \perp \cdots \perp f_k$$

such that $(V_i, f_i)$ is indecomposable.

We can take

$$f_c = (f_1)_c \perp (f_2)_c \perp \cdots \perp (f_k)_c.$$

# Indecomposable cases

| | $f$ | $f_c$ |
|---|---|---|
| (1) | $\begin{bmatrix} 0 & I_m \\ J_m(\lambda) & 0 \end{bmatrix},\ \lambda \neq 0,\ (-1)^{m+1}$ | $\begin{bmatrix} 0 & I_m \\ \lambda I_m & 0 \end{bmatrix}$ |
| (2) | $\begin{bmatrix} & & 1 \\ & -1 & -1 \\ 1 & 1 & \end{bmatrix}$ | $\begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$ |
| | $\begin{bmatrix} & & & -1 \\ & & 1 & 1 \\ & -1 & -1 & \\ 1 & 1 & & \end{bmatrix}$ | $\begin{bmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}$ |
| | $\Gamma_n,\ n \geq 1$ | |
| (3) | $J_n(0),\ n = 2m$ $\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix}$ | $\left[\begin{array}{cc\|cc} 0 & 0 & & \\ 1 & 0 & & \\ \hline & & 0 & 0 \\ & & 1 & 0 \end{array}\right]$ |

$$
f = \left[\begin{array}{ccc|ccc}
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1 \\
\hline
\lambda & & & & & \\
1 & \lambda & & & & \\
 & 1 & \lambda & & &
\end{array}\right], \quad
X = \left[\begin{array}{ccc|ccc}
1 & & & & & \\
 & t^{-1} & & & & \\
 & & t^{-2} & & & \\
 & & & 1 & & \\
 & & & & t & \\
 & & & & & t^2
\end{array}\right]
$$

$$
XfX' = \left[\begin{array}{ccc|ccc}
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1 \\
\hline
\lambda & & & & & \\
t & \lambda & & & & \\
 & t & \lambda & & &
\end{array}\right], \quad
f_c = \left[\begin{array}{ccc|ccc}
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1 \\
\hline
\lambda & & & & & \\
 & \lambda & & & & \\
 & & \lambda & & &
\end{array}\right]
$$

$$
\Gamma_3 = \left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & -1 & -1 \\
1 & 1 & 0
\end{array}\right], \quad
X = \left[\begin{array}{ccc}
t^{-1} & & \\
 & 1 & \\
 & & t
\end{array}\right]
$$

$$
X\Gamma_3X' = \left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & -1 & -t \\
1 & t & 0
\end{array}\right], \quad
\Gamma_3^+ = \left[\begin{array}{ccc}
 & & 1 \\
 & -1 & \\
1 & &
\end{array}\right]
$$

$$
f = \Gamma_n \qquad , \qquad f_c = f^{(-)^{n+1}}.
$$

The matrix of the form $f_0$

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} & & 1 \\ & 0 & 1 \\ \hline -1 & 0 & \end{array}\right]$$

$$\left[\begin{array}{cc|c|c} & & & 1 \\ & & 0 & 1 \\ \hline & 0 & 1 & \\ \hline -1 & 0 & & \end{array}\right]$$

$$\left[\begin{array}{ccc|cc} & & & 0 & 1 \\ & & & 1 & 1 \\ & & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & & \\ -1 & 0 & 0 & & \end{array}\right]$$

$$\left[\begin{array}{ccc|c|cc} & & & & 0 & 1 \\ & & & & 1 & 1 \\ & & & & 1 & 0 \\ \hline & & & 1 & & \\ \hline 0 & -1 & 0 & & & \\ -1 & 0 & 0 & & & \end{array}\right]$$

$$\left[\begin{array}{cccc|ccc} & & & & 0 & 0 & 1 \\ & & & & 0 & 1 & 1 \\ & & & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & & & \\ 0 & -1 & 0 & 0 & & & \\ -1 & 0 & 0 & 0 & & & \end{array}\right]$$

The orbit $SL_n \cdot f_0 \subset \mathcal{N}$ which is open in $\mathcal{N}$.

(1) $n = 2m + 1$

$$J_n(0) \in SL_n \cdot f_0$$

(2) $n = 2m$

$$J_n(0) \notin \mathcal{N}$$

$$J_{n-1}(0) \oplus [1] \in SL_n \cdot f_0$$

Weierstrass section of

$\pi : \mathcal{B} \to \mathcal{B}/\!/SL_n \cong K^{m+1}$

$$\left[\begin{array}{cc} \xi & \eta \\ 0 & 1 \end{array}\right] \qquad (\xi - \eta^2/4, \eta/2)$$

$$\left[\begin{array}{cc|c} \xi & \eta & 1 \\ 0 & 0 & 1 \\ \hline -1 & 0 & 0 \end{array}\right] \qquad (-\xi/4, \xi/4 - \eta)$$

$$\left[\begin{array}{cc|c|c} \xi & \eta & \zeta & 1 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} (-\xi/4 + \zeta^2/16, \\ \xi/4 - \eta - \zeta^2/8, \zeta/4) \end{array}$$

$$\left[\begin{array}{ccc|cc} \xi & \eta & \zeta & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} (\xi/16, -\xi/8 + \eta/4 + \zeta/4, \\ \xi/16 - \eta/4 + 3\zeta/4) \end{array}$$

**Main Theorem**

(1) $\mathcal{N}$ is irreducible.

(2) Each fiber $\pi^{-1}(\pi(f))$ is irreducible.

(3) Any nonzero fiber of $\pi$ contains only finitely many $SL_n$-orbits. (The number of these orbits is bounded.)

Proof based on the paper: V.L. Popov, The cone of Hilbert nullforms, *Proc. Steklov Inst. Math.* **241** (2003), 177–194.

Intersection of an orthogonal
and a symplectic group

Let $f \in \mathcal{B}$ with both $f^+$ and $f^-$ nondegenerate. Then $n = 2m$.

$$\begin{aligned} (GL_n)_f &= (GL_n)_{f^+} \cap (GL_n)_{f^-} \\ &= O(f^+) \cap Sp(f^-) \\ &= SO(f^+) \cap Sp(f^-) \end{aligned}$$

**Proposition 9.1.** For $G = (GL_n)_f$, $m \leq \dim G \leq m^2$.

(1) $\dim G = m \Rightarrow G$ is a direct product of a torus and an abelian unipotent group.

(2) $\dim G = m^2 \Rightarrow G \cong GL_m$.

| Linear operators<br>$\mathcal{E}$ | Bilinear forms<br>$\mathcal{B}$ |
|---|---|
| $u \in \mathcal{E}$ semisimple<br>iff<br>$SL_n \cdot u$ is closed | $f \in \mathcal{B}$ semisimple<br>means that<br>$SL_n \cdot f$ is closed |
| $u \in \mathcal{E}$ nilpotent | $f \in \mathcal{N}$<br>is a null-form |
| Jordan decomposition<br>$u = u_s + u_n$<br>$u_s$ semisimple<br>$u_n$ nilpotent<br>$u_s u_n = u_n u_s$<br>Decomposition is unique | $f = f_s + f_n$<br>$f_s$ semisimple<br>$f_n$ null-form<br>Decomposition is not unique! |

**Problem.** Is there a simple condition that $f_s$ and $f_n$ should satisfy to make the decomposition unique?

# Jordan Decomposition of Bilinear Forms

D.Ž. Djoković
K. Zhao

$f \in \mathcal{B}$

$Z(f) := (GL_n)_f$ centralizer of $f$

$T_f := s\ell_n \cdot f$ tangent space

to $\mathcal{O}_f := SL_n$ at $f$.

$g \in \mathcal{B}$ semisimple

$\rightarrow Z(g)$ reductive

$$\mathcal{B} = T_g \bigoplus N$$

$N = Z(g)$ – invariant subspace

$\mathcal{N}_g \subset N$ the null–cone for

the action of

$$Z(g) \cap SL_n = (SL_n)_g$$

on $N$.

## Definition of Jordan decomposition

$f \in \mathcal{B}$, $f = g + h$

(J1) $g \in \mathcal{C}_f$

$\mathcal{C}_f :=$ the unique closed $SL_n$–orbit

in $\overline{\mathcal{O}_f}$.

(J2) $h \in \mathcal{N}_g$ for some

choice of $N$,

$\mathcal{B} = T_g \oplus N$

($N$ is $Z(g)$– invariant)

(J3) $Z(f) = Z(g) \cap Z(h)$.

$K =$ an algebraically closed field

char $\quad K = 0$

$P^1K = K \cup \{\infty\}$, projective line

$\iota : P^1K \to P^1K$

$\iota(\lambda) = \lambda^{-1}$.

$Z_2 = \langle \iota \rangle$

$\Pi = P^1K/Z_2$

Definition: $\mu : P^1K \to P^1K$

is a $\iota -$ map if

1) $\iota\mu = \mu\iota$

2) $\mu(1) = -1\,,\ \mu(-1) = 1$.

To each $\iota$–map, $\mu$, we

associate a Jordan decomposition

$f = g_\mu + h_\mu$

**Primary Decomposition**

For $f \in \mathcal{B} \backslash \mathcal{N}$ there is a unique decomposition

$$(V, f) = \perp_{\pi \in \Pi} (V^\pi, f^\pi).$$

such that

$V^{\{0,\infty\}}$ is totally degenerate

and otherwise

$(V^\pi, f^\pi)$ is nondegenerate

and $\pi$ is the set of eigenvalues

of the asymmetry $\sigma^\pi$ of $f^\pi$.

**Theorem**

(a) $f \in \mathcal{B}\backslash\mathcal{N}$ , $f = g + h$

a Jordan decomposition,

$g$ semisimple.

Then $f$ and $g$ have the

same primary decompositons

$$V = \perp_{\pi \in \Pi} V^{\pi} \ .$$

$$f^{\pi} = g^{\pi} + h^{\pi}$$

is a Jordan decomposition $\forall \pi \in \Pi$.

(b) Converse holds.

**Primary Case**

$\pi \in \Pi,\ \pi = \{\lambda, \lambda^{-1}\},\ \lambda \neq \lambda^{-1}.$

$f \notin \mathcal{N},\ f = f^{\pi}.$

Define $\varphi := \frac{f - \lambda f'}{1 - \lambda}.$

Then

$$\varphi \notin \mathcal{N}, \quad \varphi \text{ is } \{0, \infty\}\text{-primary.}$$

$L(W) = \{v \in V : \varphi(v, W) = 0\}$

$R(W) = \{v \in V : \varphi(W, v) = 0\}$

$L(V) \subset L^3(V) \subset L^5(V) \subset \cdots \subset L_\infty(\varphi)$

$R(V) \subset R^3(V) \subset R^5(V) \subset \cdots \subset R_\infty(\varphi)$

$V_\lambda(f) := L_\infty(\varphi),\ V_{1/\lambda}(f) := R_\infty(\varphi)$

$V = V_{1/\lambda}(f) \oplus V_\lambda(f)$

$f = g_\mu + h_\mu$

$\mu$-Jordan decomposition

$\mu =$ a $\iota$-map

$$g_\mu = \begin{bmatrix} 0 & \frac{f' - \mu(\lambda) f}{\lambda - \mu(\lambda)} \\ \lambda \cdot \frac{f - \mu(\lambda) f'}{\lambda - \mu(\lambda)} & 0 \end{bmatrix}$$

$$h_\mu = \begin{bmatrix} 0 & \frac{\lambda f - f'}{\lambda - \mu(\lambda)} \\ \mu(\lambda) \cdot \frac{\lambda f' - f}{\lambda - \mu(\lambda)} & 0 \end{bmatrix}$$

$g_\mu, h_\mu$ are independent of the choice of $\lambda \in \pi$.

If $\pi \neq \{0, \infty\}$, then $V_\lambda(f)$ is the generalized $\lambda$-eigenspace of the asymmetry of $f$.

Three special $\iota$-maps:

1) Jordan decomposition of symmetric type

$$\begin{aligned} \mu_+(\lambda) &= 1 && \text{if } \lambda \neq 1 \\ &= -1 && \text{if } \lambda = 1 \end{aligned}$$

2) Jordan decomposition of skew-symmetric type

$$\begin{aligned} \mu_-(\lambda) &= -1 && \text{if } \lambda \neq -1 \\ &= 1 && \text{if } \lambda = -1 \end{aligned}$$

3) Jordan decomposition of harmonic type

$$\begin{aligned} \mu_0(\lambda) &= \lambda^{-1} && \text{if } \lambda \neq \pm 1 \\ &= -1 && \text{if } \lambda = 1 \\ &= 1 && \text{if } \lambda = -1 \end{aligned}$$