Representation type of Hecke algebras (Resume)

Susumu Ariki

Let A be a finite dimensional algebra over an algebraically closed field F. The representation type of A is an index for the possibility to classify simple A-modules.

- If A is of finite type, then we can classify simple A-modules in principle, as the number is finitely many.
- If A is of tame type, we can classify simple A-modules of given dimension, as the modules are parametrized by a finite number of ≤ 1 parameter families.
- If A is of wild type, the classification is hopeless, as the problem contains the classification problem of any finite dimensional algebra.

The actual classification is a different problem, and could be difficult. Our aim is to explain that the problem can be handled when the algebra is a Hecke algebra.

Let W be a finite Weyl group of classical type, i.e. the irreducible components are of type A, BC or D, and let $q \in F$ be invertible. We denote by $\mathcal{H}_W(q)$ the associated Hecke algebra. If $q = 1$ then $\mathcal{H}_W(q) = FW$ and we are reduced to the case of group algebras. We are interested in the case where $q \neq 1$. Before stating our result, we introduce the Poincaré polynomial.

Definition 1. *Let $\ell(w)$ be the length of $w \in W$. The Poincaré polynomial of W is defined by*

$$P_W(x) = \sum_{w \in W} x^{\ell(w)}.$$

Theorem 2 (A). *Suppose that W, q and $\mathcal{H}_W(q)$ be as above. Then $\mathcal{H}_W(q)$ is*

- *finite if $(x-q)^2$ does not divide $P_W(x)$.*
- *tame if $q = -1 \neq 1$ and $(x+1)^2 || P_W(x)$.*
- *wild in the remaining cases.*

A corollary of the proof is the following.

Corollary 3. *Suppose that W, q and $\mathcal{H}_W(q)$ be as above and let B be a block algebra of $\mathcal{H}_W(q)$.*

(1) *If B is of finite type then B is a Brauer tree algebra with tree being a straight line without exceptional vertex.*
(2) *If B is of tame type then B is a special biserial algebra.*

If B is as in (1) then we are essentially in the case of irreducible Weyl groups, and PIM's are of the form

$$\begin{matrix} D_1 & D_i & & D_r \\ D_2, & D_{i-1}\oplus D_{i+1} & \text{for } 2\le i\le r-1, & D_{r-1}. \\ D_1 & D_i & & D_r \end{matrix}$$

The number of simple B-modules is given by the following formula.

(i) $r=o(q)-1$ if W is type A or if $o(q)$ is odd.

(ii) $r=\frac{o(q)}{2}$ or $\frac{o(q)}{2}+2$ if W is type B and $o(q)$ is even.

(iii) $r=\frac{o(q)}{2}+1$ if W is type D and $o(q)$ is even.

Here, $o(q)$ is the multiplicative order of q.

Now we remove the assumptions on $\mathcal{H}_W(q)$. We conjecture that every block algebra of finite type is a Brauer tree algebra with tree being a straight line without exceptional vertex, and every block algebra of tame type is a special biserial algebra.

This implies that we can classify all the simple modules if B is not of wild type:

- if B is a Brauer tree algebra of this type, the number of indecomposable modules is $r(r+1)$ and we can describe them explicitly.
- if B is a special biserial algebra and self-injective, it is well known that the stable Auslander-Reiten quiver of B is the Auslander-Reiten quiver of $B/\operatorname{Soc} B$. As the latter is a string algebra, the indecomposable modules are given by string modules and band modules, which are given by colored quivers of type A and $\tilde{A}$ whose edges are colored by the arrows of the algebra.

Reference: Theorem(A) is proved in math.QA/0302136

S. Ariki, Hecke algebras of classical type and their representation type.

RIMS, Kyoto university, Kyoto 606–8502, Japan
E-mail address: ariki@kurims.kyoto-u.ac.jp