Theta lifting of the trivial representation and the associated nilpotent orbit
— the case of $U(p,q) \times U(n,n)$ — *

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Contents

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Introduction

In [10], Ochiai, Taniguchi and one of the authors calculate the associated cycles and Bernstein degrees for certain singular unitary representations. Their method uses the theory of dual pairs (or theta correspondence), though it is only applicable to pairs of compact type (i.e., one member of the pair is a compact group). In this note, we consider representations arising from a dual pair of non-compact type, and our purpose is to establish a generalization of the results in [10].

One problem we encounter here is that the theta correspondence for a non-compact pair is not completely understood, in spite of the pioneer work of Howe, and subsequently excellent works of many people such as Adams, Li, Moeglin, Przebinda. Another problem here is the lack of information on the unitary representations of non-compact groups which we try to lift. In the case of compact Lie groups, unitary representations are well-understood and we can do many computations in full details thanks to Cartan-Weyl theory and Frobenius reciprocity. For non-compact groups, such information on unitary representations are not readily available, let alone their classifications.

Therefore, in this note, we shall restrict ourselves to theta lifts of well-known unitary representations such as the trivial representation and holomorphic discrete series representations. To present the ideas clearly, we shall also fix our dual pair as

$$U(p,q) \times U(n,n) \quad (p+q \leq n).$$

The condition $p+q \leq n$ ensures that the pair is in the stable range with $U(p,q)$ as the smaller member. We take an irreducible unitary representation of the smaller group $U(p,q)$ (to be more precise, the metaplectic covering group of $U(p,q)$), and investigate its theta lift. In the cases we consider, it turns out that the associated variety of the theta lift corresponds to that of the original representation through a "complex moment map", as expected. Moreover, the associated cycles seem to behave well under this correspondence. These examples strongly suggest that the conjectured formula of associated cycles in [10] is true in general.

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1 Some small nilpotent orbits in $U(n,n)$

Let $G = U(n,n) \supset K = U(n) \times U(n)$, with $G_{\mathbb{C}} = GL_{2n}$ and $K_{\mathbb{C}} = GL_n \times GL_n$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the complexified Lie algebras of G and K respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition, and $\mathfrak{s} = \mathfrak{s}^+ \oplus \mathfrak{s}^-$ be the $K_{\mathbb{C}}$-stable decomposition of $\mathfrak{s}$.

We begin with some elementary linear algebra. Write a typical element of $\mathfrak{s}$ as

$$X = \left[\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right] \in \mathfrak{s},$$

where $A \in \mathfrak{s}^+ = M_n(\mathbb{C}),\ B \in \mathfrak{s}^- = M_n(\mathbb{C})$.

Since

$$X^2 = \left[\begin{array}{c|c} AB & \\ \hline & BA \end{array}\right],$$

we see that X is nilpotent if and only if AB and BA are both nilpotent. In fact, this is equivalent to say that AB or BA is nilpotent.

Note that for $(g, h) \in K_{\mathbb{C}} = GL_n \times GL_n$, we have

$$\mathrm{Ad}\,(\mathrm{diag}\,(g, h)) \left[\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & gAh^{-1} \\ \hline hBg^{-1} & 0 \end{array}\right] \overset{put}{=} \left[\begin{array}{c|c} 0 & A' \\ \hline B' & 0 \end{array}\right].$$

Thus the $K_{\mathbb{C}}$-orbit through X, denoted by $\mathcal{O}_X$, has invariants $\operatorname{rank} A$ and $\operatorname{rank} B$. Further we have

$$\begin{cases} A'B' = gABg^{-1}, \\ B'A' = hBAh^{-1}. \end{cases}$$

Therefore AB and BA naturally define GL_n-conjugacy classes in $M_n(\mathbb{C})$.

Let $\mathfrak{X}_{\mathrm{null}}$ be the cone consisting of those nilpotent elements X in $\mathfrak{s}$ such that $X^2 = 0$, namely

$$\mathfrak{X}_{\mathrm{null}} = \left\{ X = \left[\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right] \in \mathfrak{s} \mid AB = 0, BA = 0 \right\}.$$

Put

$$x_{p,q} = \left[\begin{array}{c|c} & \begin{matrix} 1_p & \\ & 0 \end{matrix} \\ \hline \begin{matrix} 0 & \\ & 1_q \end{matrix} & \end{array}\right] \in \mathfrak{X}_{\mathrm{null}} \quad (p + q \leq n). \tag{1.1}$$

Denote by $\mathcal{O}_{p,q} = K_{\mathbb{C}} \cdot x_{p,q}$ the adjoint $K_{\mathbb{C}}$-orbit through $x_{p,q}$.

Lemma 1.1 (1) *We have*

$$\overline{\mathcal{O}_{p,q}} \subset \mathfrak{X}_{\mathrm{null}},$$

where $\overline{\mathcal{O}_{p,q}}$ *denotes the closure of the orbit* $\mathcal{O}_{p,q}$*. Moreover, we have the decomposition*

$$\mathfrak{X}_{\mathrm{null}} = \coprod_{p+q \leq n} \mathcal{O}_{p,q}.$$

(2) $\mathcal{O}_{p,q}$ *and* $\mathcal{O}_{r,s}$ *generate the same* $G_{\mathbb{C}}$*-orbit if and only if* $p + q = r + s$*. Moreover, we have*

$$\dim \mathcal{O}_{p,q} = (p + q)(2n - (p + q)), \quad \overline{\mathcal{O}_{p,q}} = \coprod_{p' \leq p,\ q' \leq q} \mathcal{O}_{p',q'}.$$

PROOF. (1) It is clear that $\mathcal{O}_{p,q} \subset \mathfrak{X}_{\text{null}}$ and $\mathfrak{X}_{\text{null}}$ is closed. Hence we have $\overline{\mathcal{O}_{p,q}} \subset \mathfrak{X}_{\text{null}}$. Now take an arbitrary element of $\mathfrak{X}_{\text{null}}$. By replacing it by a $K_{\mathbb{C}}$-conjugate, we can assume that the element is of the form

$$\left[\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right] \in \mathfrak{X}_{\text{null}}, \quad A = \begin{bmatrix} 1_p & \\ & 0 \end{bmatrix}.$$

By the assumption that $AB = 0, BA = 0$, we must have

$$B = \begin{bmatrix} 0 & \\ & B' \end{bmatrix}.$$

Since elements of the form

$$(g,h) = \left(\begin{bmatrix} 1_p & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 1_p & 0 \\ 0 & * \end{bmatrix} \right) \in K_{\mathbb{C}}$$

fix A, we can conjugate B by some (g,h) to obtain $B' = \operatorname{diag}(0, 1_q)$.

(2) It is obvious that the Jordan form of $x_{p,q}$ depends only on $p+q$. This implies that $\mathcal{O}_{p,q}$ and $\mathcal{O}_{r,s}$ generate the same $G_{\mathbb{C}} = GL_{2n}$-orbit if and only if $p+q = r+s$. To calculate the dimension of $\mathcal{O}_{p,q}$, we consider the fixed subgroup of $x_{p,q}$ in $K_{\mathbb{C}}$, denoted by $K_{p,q}$. We have

$$K_{p,q} = \{(g,h) \in K_{\mathbb{C}} \mid g \text{ and } h \text{ are of the following form}\}, \tag{1.2}$$

$$g = \begin{bmatrix} a & * & * \\ 0 & * & * \\ 0 & 0 & c \end{bmatrix}, \quad h = \begin{bmatrix} a & 0 & 0 \\ * & * & 0 \\ * & * & c \end{bmatrix}; \quad a \in GL_p, c \in GL_q.$$

The dimension formula then follows. Finally we observe that an orbit in $\mathfrak{X}_{\text{null}}$ is determined by $\operatorname{rank} A$ and $\operatorname{rank} B$ (see the proof of (1)), we therefore conclude that the closure $\overline{\mathcal{O}_{p,q}}$ contains all the low rank matrices. Q.E.D.

Proposition 1.2 *Assume that $p+q < n$.*

(1) *The variety $\overline{\mathcal{O}_{p,q}}$ is normal.*

(2) *The ring of regular functions on $\mathcal{O}_{p,q}$ and $\overline{\mathcal{O}_{p,q}}$ coincide:*

$$\mathbb{C}[\mathcal{O}_{p,q}] = \mathbb{C}[\overline{\mathcal{O}_{p,q}}] \quad (p+q<n).$$

Remark 1.3 If $p+q=n$, we will see that $\mathbb{C}[\mathcal{O}_{p,q}] \neq \mathbb{C}[\overline{\mathcal{O}_{p,q}}]$ afterwards (see Lemma 1.4).

PROOF. We postpone the proof of the normality of $\overline{\mathcal{O}_{p,q}}$ to §4 (Corollary 4.7).

(2) We calculate the codimension of the boundary component $\overline{\mathcal{O}_{p,q}} \setminus \mathcal{O}_{p,q}$. Put $r = p+q$. Then this is equal to

$$r(2n-r) - (r-1)(2n-(r-1)) = 2(n-r)+1.$$

Therefore, if $r < n$, the codimension is ≥ 2. This implies (2) under the statement (1). Q.E.D.

We denote by Λ_n^+ the set of all dominant integral weights of GL_n, namely

$$\Lambda_n^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\},$$

and

$$\mathcal{P}_n = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\},$$

the set of all partitions of length $\leq n$. For $\lambda \in \Lambda_n^+$, $\tau_\lambda^{(n)}$ denotes the irreducible finite dimensional representation of GL_n of highest weight λ (with respect to the upper triangular Borel subgroup B of GL_n).

For $p + q \leq n$, we put

$$\Lambda^+(p, q) = \{(\alpha, 0, \dots, 0, \gamma^*) \in \mathbb{Z}^n \mid \alpha \in \mathcal{P}_p, \gamma \in \mathcal{P}_q\}, \tag{1.3}$$

where $\gamma^* = (-\gamma_q, \dots, -\gamma_2, -\gamma_1)$, for $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_q)$. We denote $\lambda = (\alpha, 0, \dots, 0, \gamma^*)$ by $\alpha \odot \gamma$.

Lemma 1.4 *The ring of regular functions on $\mathcal{O}_{p,q}$ (or $\overline{\mathcal{O}_{p,q}}$), denoted by $\mathbb{C}[\mathcal{O}_{p,q}]$ (or $\mathbb{C}[\overline{\mathcal{O}_{p,q}}]$), inherits the action of $K_\mathbb{C} = GL_n \times GL_n$.*

(1) If $p + q < n$, then we have

$$\mathbb{C}[\overline{\mathcal{O}_{p,q}}] = \mathbb{C}[\mathcal{O}_{p,q}] \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} (\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)} \quad (\text{as } K_\mathbb{C}\text{-module}).$$

(2) If $p + q = n$, then we have

$$\mathbb{C}[\mathcal{O}_{p,q}] \simeq \sum_{\lambda \in \Lambda_n^+}^{\oplus} (\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)},$$

while

$$\mathbb{C}[\overline{\mathcal{O}_{p,q}}] \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} (\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)}.$$

Proof. Recall the fixed subgroup $K_{p,q}$ of $x_{p,q}$ in $K_\mathbb{C}$ (cf. (1.2)). We have

$$\mathbb{C}[\mathcal{O}_{p,q}] \simeq \mathbb{C}[K_\mathbb{C}]^{K_{p,q}} \simeq \operatorname{Ind}_{K_{p,q}}^{K_\mathbb{C}} \mathbf{1}_{K_{p,q}}.$$

By the Frobenius reciprocity of algebraic groups (or from the second term of the above expression), $\pi = \tau_\mu^{(n)} \boxtimes \tau_\lambda^{(n)} \in \operatorname{Irr}(K_\mathbb{C})$ occurs in $\mathbb{C}[\mathcal{O}_{p,q}]$ if and only if

$$\operatorname{Hom}_{K_{p,q}}(\pi, \mathbf{1}) \neq 0.$$

We consider intermediate parabolic $P \times \overline{P} \supset K_{p,q}$, where

$$P = \begin{bmatrix} GL_p & * & * \\ 0 & GL_{n-(p+q)} & * \\ 0 & 0 & GL_q \end{bmatrix}, \quad \overline{P} = (\text{opposite of } P)$$

with the Levi-decompositions $P = MN$ and $\overline{P} = M\overline{N}$.

We can write $K_{p,q} = L_{p,q}N_{p,q}$, where $L_{p,q} \simeq GL_p \times GL_{n'} \times GL_{n'} \times GL_q$ $(n' = n-(p+q))$ and $N_{p,q} = N \times \overline{N}$. Here the subgroup $GL_p \times GL_q$ of $L_{p,q}$ embeds in $M \times M$ diagonally. We have

$$\operatorname{Hom}_{K_{p,q}}(\pi, \mathbf{1}) \simeq \operatorname{Hom}_{L_{p,q}}(\tilde{\pi}, \mathbf{1}),$$

where

$$\tilde{\pi} = \tau_\lambda^{(n)}/\mathfrak{n}\tau_\mu^{(n)} \boxtimes \tau_\lambda^{(n)}/\overline{\mathfrak{n}}\tau_\mu^{(n)},$$

and $\mathfrak{n}$, $\overline{\mathfrak{n}}$ are the Lie algebras of N, $\overline{N}$, respectively.

Note that $\tilde{\pi}$ is irreducible under the action of $M \times M$. since $GL_p \times GL_q$ embeds in $M \times M$ diagonally, we see that

$$\operatorname{Hom}_{L_{p,q}}(\tilde{\pi}, \mathbf{1}) \neq 0$$

if and only if $\tilde{\pi}$ is of the form

$$(\tau_\alpha^{(p)} \boxtimes \mathbf{1}_{GL_{n'}} \boxtimes \tau_\beta^{(q)}) \boxtimes (\tau_{\alpha'}^{(p)} \boxtimes \mathbf{1}_{GL_{n'}} \boxtimes \tau_{\beta'}^{(q)})$$

with $\alpha' = \alpha^*$ and $\beta' = \beta^*$, and with the obvious notations.

The highest weight of $\tau_\mu^{(n)}$ with respect to B is $\lambda = (\alpha, 0, \ldots, 0, \beta)$, and the highest weight of $\tau_\lambda^{(n)}$ with respect to $\overline{B}$ is $(\alpha^*, 0, \ldots, 0, \beta^*)$. Thus $\lambda = \mu^*$, and it is dominant. If $p+q < n$, this means that $\alpha \in \mathcal{P}_p$ and $\beta = \gamma^*$ for some $\gamma \in \mathcal{P}_q$. If $p+q = n$, $\lambda = (\alpha, \beta)$ is a dominant weight if $\alpha_1 \geq \cdots \geq \alpha_p \geq \beta_1 \geq \cdots \geq \beta_q$ ($\alpha_p \geq \beta_1$ is the essential restriction). This proves (1) and the first half of (2) (cf. Proposition 1.2).

Next we assume that $p+q = n$, and consider the second half of (2). We shall prove this by explicitly constructing a highest weight vector for $(\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)}$ $(\lambda \in \Lambda_n^+)$ which should be unique up to a scalar. For

$$X = \left[\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right],$$

put

$$\begin{cases} d_k^+(X) &= (\text{upper principal minor of } A \text{ of size } k) \\ d_k^-(X) &= (\text{lower principal minor of } B \text{ of size } k) \end{cases}.$$

Then a highest weight vector of $(\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)}$ $(\lambda = \alpha \odot \gamma)$ is given by

$$d(\alpha, \gamma) = \left(\prod_{k=1}^{p-1} (d_k^+)^{\alpha_k - \alpha_{k+1}} \right) (d_p^+)^{\alpha_p} \cdot \left(\prod_{j=1}^{q-1} (d_j^-)^{\gamma_j - \gamma_{j+1}} \right) (d_q^-)^{\gamma_q}.$$

This function extends to $\overline{\mathcal{O}_{p,q}}$ if and only if $\alpha_p \geq 0$ and $\gamma_q \geq 0$. Q.E.D.

We shall give an alternative proof of the above lemma (at least for $\mathbb{C}[\overline{\mathcal{O}_{p,q}}]$) in §4 (see Theorem 4.6).

2 Theta lift of the trivial representation of $U(p,q)$

We reinterpret one of the results of Lee and Zhu [5] in a naive and intuitive way. First consider a seesaw pair:

$$\begin{array}{rcccl} M' = & U(p,q) \times U(p,q) & & U(n,n) & = G \\ & \text{diagonal} \quad \cup & \times & \cup & \\ G' = & U(p,q) & & U(n) \times U(n) & = K \end{array}$$

We still keep to the condition $p+q \leq n$. We consider the dual pair $U(p,q) \times U(n)$ of compact type in $Sp(2n(p+q), \mathbb{R})$. By the $U(p,q) \times U(n)$ duality, we have the decomposition of the Weil representation ω of the metaplectic double cover $Mp(2n(p+q), \mathbb{R})$:

$$\omega \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} L(\tau_\lambda^{(n)}) \boxtimes (\tau_\lambda^{(n)} \otimes \chi_{p,q}) \quad (\text{as a } U(p,q)^\sim \times U(n)^\sim\text{-module}), \tag{2.1}$$

where $H^\sim$ denotes the inverse image of H in the metaplectic cover $Mp(2n(p+q), \mathbb{R})$, and $\chi_{p,q} = \det^{(p-q)/2}$. If $p+q \leq n$, the irreducible representation $L(\tau_\lambda^{(n)})$ is a holomorphic discrete series of $U(p,q)^\sim$ with minimal K-type $(\tau_\alpha^{(p)} \otimes \det^{n/2}) \boxtimes (\tau_\gamma^{(q)} \otimes \det^{n/2})^*$ where $\lambda = \alpha \odot \gamma \in \Lambda^+(p,q)$.

Now consider two $U(p,q) \times U(n)$-dualities, one holomorphic and the other anti-holomorphic (or contragredient), and make a tensor product on them:

$$\omega \otimes \omega^* \simeq \left(\sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} L(\tau_\lambda^{(n)}) \boxtimes (\tau_\lambda^{(n)} \otimes \chi_{p,q}) \right) \otimes \left(\sum_{\mu \in \Lambda^+(p,q)}^{\oplus} L(\tau_\mu^{(n)})^* \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})^* \right)$$

$$\simeq \sum_{\lambda,\mu}^{\oplus} \left(L(\tau_\lambda^{(n)}) \otimes L(\tau_\mu^{(n)})^* \right) \big|_{\Delta U(p,q)^\sim} \boxtimes ((\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})^*). \tag{2.2}$$

The last decomposition is the one as a representation of $U(p,q)^\sim \times (U(n)^\sim \times U(n)^\sim)$. For the above decompositions, we refer the readers to [3] and [4]. Also the reference to [10] will be helpful.

Lemma 2.1 *There exists a non-zero quotient map*

$$L(\tau_\lambda) \otimes L(\tau_\mu)^* \to \mathbf{1}_{U(p,q)^\sim} : \text{ surjective}$$

if and only if $\lambda = \mu$. *In that case,*

$$\dim \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K}')}(L(\tau_\lambda) \otimes L(\tau_\lambda)^*, \mathbf{1}_{U(p,q)^\sim}) = 1,$$

where $\mathfrak{g}' = \mathfrak{u}(p,q)$ *and* $K' = U(p) \times U(q)$.

Proof. We shall prove this in a more general setting afterwards. See Lemma 5.1. Q.E.D.

Corollary 2.2 *Howe's maximal quotient* $\Omega^{p,q}(\mathbf{1}_{U(p,q)^\sim})$ *has the* K*-type decomposition*

$$\Omega^{p,q}(\mathbf{1}_{U(p,q)^\sim})\big|_{\widetilde{K}} \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} (\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\lambda^{(n)} \otimes \chi_{p,q})^*.$$

Proof. Howe's maximal quotient is canonically isomorphic to the dual of

$$\operatorname{Hom}_{(\mathfrak{g}', \widetilde{K}')}(\omega \otimes \omega^*, \mathbf{1}_{U(p,q)^\sim}).$$

Then the above Lemma 2.1 and (2.2) give the K-type decomposition. Q.E.D.

In the following, we write $\Omega^{p,q}(\mathbf{1})$ instead of $\Omega^{p,q}(\mathbf{1}_{U(p,q)^\sim})$.

Theorem 2.3 (Lee-Zhu) *The maximal quotient* $\Omega^{p,q}(\mathbf{1})$ *is irreducible and gives the theta-lift* $\theta^{p,q}(\mathbf{1})$. *Thus we get*

$$\theta^{p,q}(\mathbf{1})\big|_{\widetilde{K}} \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} (\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\lambda^{(n)} \otimes \chi_{p,q})^*.$$

Proof. It is known that, in the stable range, the maximal quotient of the trivial representation is irreducible, and coincides with the theta lift ([13, Proposition 2.1]; see also [6]). Then the theorem follows from the above consideration. This proof is an alternative of that of Lee and Zhu ([5]). Q.E.D.

Recall the identification of nilpotent orbits in $\mathfrak{g}$ with the nilpotent coadjoint orbits in $\mathfrak{g}^*$. Note that the associated variety is by definition an object in the dual space $\mathfrak{g}^*$. Therefore, to consider the associated variety of $\theta^{p,q}(\mathbf{1})$, we should take the dual orbit of $\overline{\mathcal{O}_{p,q}}$ with respect to the Killing form, which is $\overline{\mathcal{O}_{q,p}}$ as a result. Since the $K_{\mathbb{C}}$-module structure on $\mathbb{C}[\overline{\mathcal{O}_{q,p}}]$ is the contragredient of that of $\mathbb{C}[\overline{\mathcal{O}_{p,q}}]$, Theorem 2.3 together with Lemma 1.4 strongly suggests that the associated variety of the theta lift $\theta^{p,q}(\mathbf{1})$ coincides with $\overline{\mathcal{O}_{q,p}}$. This is in fact true.

Theorem 2.4 *The associated cycle of* $\theta^{p,q}(\mathbf{1})$ *is* $[\overline{\mathcal{O}_{q,p}}]$ *without multiplicity. In particular, we have* $\operatorname{Deg}\theta^{p,q}(\mathbf{1}) = \deg\overline{\mathcal{O}_{q,p}}$*, where* $\operatorname{Deg}\theta^{p,q}(\mathbf{1})$ *denotes the Bernstein degree of* $\theta^{p,q}(\mathbf{1})$*.*

Remark 2.5 The result of Przebinda et al. [2], [11], [12] determines the asymptotic support of $\theta^{p,q}(\mathbf{1})$. From the result of Schmid and Vilonen, the asymptotic support and the associated variety correspond to each other through Kostant-Sekiguchi correspondence. Therefore in principle we can compute the associated variety of $\theta^{p,q}(\mathbf{1})$ from the known result of Przebinda et al.

We shall give the proof in section 4.

Similar method can be used to compute the K-type decomposition of the theta lift of unitary characters of $U(p,q)^{\sim}$, and their associated cycles.

3 Degree of orbits $\mathcal{O}_{p,q}$

In this section, we derive an explicit formula for $\deg\overline{\mathcal{O}_{p,q}} = \deg\overline{\mathcal{O}_{q,p}} = \operatorname{Deg}\theta^{p,q}(\mathbf{1})$.

Consider $\lambda \in \Lambda^+(p,q)$. Then

$$\lambda = (\alpha_1, \dots, \alpha_p, 0, \dots, 0, -\gamma_1, \dots, -\gamma_q) \overset{\text{put}}{=} \alpha \odot \gamma \quad (\alpha \in \mathcal{P}_p, \gamma \in \mathcal{P}_q).$$

From the explicit description of the highest weight vector for $(\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)}$ (cf. (1)), we conclude that the homogeneous degree of $(\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)}$ is

$$\|\lambda\| = |\alpha| + |\gamma|.$$

Note that $\|\lambda\| \neq |\alpha| - |\gamma|$. Therefore, to get the degree of the projective variety $\overline{\mathcal{O}_{p,q}}$, we consider the following Hilbert-Samuel function

$$H(k) = \sum_{\|\lambda\| \leq k, \lambda \in \Lambda^+(p,q)} \dim(\tau_\lambda^{(n)})^* \boxtimes \tau_\lambda^{(n)} = \sum_{\|\lambda\| \leq k, \lambda \in \Lambda^+(p,q)} (\dim \tau_\lambda)^2. \tag{3.1}$$

By Weyl's dimension formula, we have

$$\dim \tau_\lambda = \prod_{1 \leq i < j \leq n} \frac{\langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle}{\langle \rho, \varepsilon_i - \varepsilon_j \rangle},$$

where $\langle \lambda, \varepsilon_i \rangle = \lambda_i$ and

$$\rho = \left(\frac{n-1}{2}, \dots, \frac{n+1}{2} - k, \dots, \frac{1-n}{2} \right).$$

The result is the following rather complicated formula (here $n' = n - (p+q)$):

$$\dim \tau_\lambda = \frac{\prod_{p+1 \le i < j \le p+n'}(j-i)}{\prod_{1 \le i < j \le n}(j-i)} \cdot$$
$$\prod_{1 \le i < j \le p} (\alpha_i - \alpha_j + j - i) \prod_{1 \le i \le p < j \le p+n'} (\alpha_i + j - i) \times$$
$$\prod_{1 \le i \le p, p+n' < j \le n} (\alpha_i + \gamma_{j-(p+n')} + j - i) \times$$
$$\prod_{p+n' < i < j \le n} (\gamma_{j-(p+n')} - \gamma_{i-(p+n')} + j - i) \prod_{p < i \le p+n' < j \le n} (\gamma_{j-(p+n')} + j - i).$$

From this formula, $H(k)$ becomes, for a sufficiently large k,

$$H(k) = \frac{k^d}{p!q!\left(\prod_{j=n'}^{n-1} j!\right)^2} \int_W \left(\prod_{i=1}^p x_i \prod_{j=1}^q y_j\right)^{2n'} \prod_{1 \le i \le p, 1 \le j \le q} (x_i + y_j)^2 \times$$
$$|\Delta(x_1, \dots, x_p)|^2 |\Delta(y_1, \dots, y_q)|^2 \prod_{i=1}^p dx_i \prod_{j=1}^q dy_j + O(k^{d-1}),$$

where Δ denotes the difference product, and

$$W = \{(x,y) \mid x_i \ge 0, y_j \ge 0, \sum_{i=1}^p x_i + \sum_{j=1}^q y_j \le 1\}, \tag{3.2}$$
$$d = (p+q)(2n - (p+q)) = n^2 - (n')^2.$$

We therefore have the following

Theorem 3.1 *Put $n' = n - (p+q)$. Then*

$$\deg \overline{\mathcal{O}_{p,q}} = \frac{(n^2 - (n')^2)!}{p!q!\left(\prod_{j=n'}^{n-1} j!\right)^2} \int_W \left(\prod_{i=1}^p x_i \prod_{j=1}^q y_j\right)^{2n'} \prod_{1 \le i \le p, 1 \le j \le q} (x_i + y_j)^2 \times$$
$$|\Delta(x_1, \dots, x_p)|^2 |\Delta(y_1, \dots, y_q)|^2 \prod_{i=1}^p dx_i \prod_{j=1}^q dy_j,$$

where W is the region given in (3.2).

We expect that we can calculate this integral by the similar method as in [9].

Example 3.2 Let us consider the simplest case which is non-trivial, i.e., the case where $p = q = 1$. Then we get

$$\deg \overline{\mathcal{O}_{1,1}} = \frac{(4(n-1))!}{\{(n-2)!(n-1)!\}^2} \int_{x+y \le 1,\ x,y \ge 0} (xy)^{2(n-2)} (x+y)^2 dx dy. \tag{3.3}$$

Lemma 3.3 *We have*

$$\int\limits_{x+y\le 1,\, x,y\ge 0} x^\alpha y^\beta (x+y)^\gamma dx dy = \frac{B(\alpha+1,\beta+1)}{\alpha+\beta+\gamma+2},$$

where $B(s,t)$ denotes the Beta function.

From this lemma, we get

$$\deg \overline{\mathcal{O}_{1,1}} = \frac{(4(n-1))!}{\{(n-2)!(n-1)!\}^2} \frac{\{(2(n-2))!\}^2}{4(n-1)(4(n-2)+1)!} = \frac{(4\,n-5)\,(\Gamma(2\,n-1))^2}{2\,(2\,n-3)\,(\Gamma(n))^4}.$$

4 Nullcone and the geometry of nilpotent orbits

In this section, we consider a dual pair $(G', G) = (U(p,q), U(r,s))$. Afterwards we will take $r = s = n$ and assume that $p + q \le n$; but we do not assume any restriction on the pair for the moment.

Let $K'_{\mathbb{C}} = GL_p \times GL_q$ and $K_{\mathbb{C}} = GL_r \times GL_s$, which are the complexifications of maximal compact subgroups $K' = U(p) \times U(q) \subset G'$ and $K = U(r) \times U(s) \subset G$. We define a $K'_{\mathbb{C}} \times K_{\mathbb{C}}$-action on

$$\mathcal{W} = M_{p+q,r+s}(\mathbb{C})$$

as follows:

$$M_{p+q,r+s} \ni T = \begin{array}{cc} \begin{array}{cc} r & s \end{array} & \\ \left[\begin{array}{c|c} X & Z \\ \hline Y & W \end{array}\right] & \begin{array}{c} p \\ q \end{array} \end{array}, \quad k = (a,b) \times (g,h) \in K'_{\mathbb{C}} \times K_{\mathbb{C}},$$

$$k \cdot T = \left[\begin{array}{c|c} aX\,{}^t g & {}^t a^{-1} Z h^{-1} \\ \hline {}^t b^{-1} Y g^{-1} & bW\,{}^t h \end{array}\right].$$

We define two $K'_{\mathbb{C}} \times K_{\mathbb{C}}$-equivariant maps

$$\begin{cases} \varphi_1 : & M_{p+q,r+s} \longrightarrow M_{r,s} \oplus M_{s,r} = \mathfrak{s}, \\ \varphi_2 : & M_{p+q,r+s} \longrightarrow M_{p,q} \oplus M_{q,p} = \mathfrak{s}' \end{cases}$$

by

$$\begin{cases} \varphi_1(T) = ({}^t XZ, {}^t WY), \\ \varphi_2(T) = (X\,{}^t Y, W\,{}^t Z). \end{cases}$$

Here $K_{\mathbb{C}}$ (resp., $K'_{\mathbb{C}}$) acts on $\mathfrak{s}$ (resp., $\mathfrak{s}'$) by the restriction of the Adjoint action, and $K'_{\mathbb{C}}$ (resp., $K_{\mathbb{C}}$) acts on $\mathfrak{s}'$ (resp., $\mathfrak{s}$) trivially.

For $D \in \mathfrak{s}$ and $D' \in \mathfrak{s}'$, we say that D corresponds to D' if there exists $T \in M_{p+q,r+s}$ such that $\varphi_1(T) = D$, and $\varphi_2(T) = D'$. Further we say a $K_{\mathbb{C}}$ orbit $\mathcal{O}$ corresponds to a $K'_{\mathbb{C}}$ orbit $\mathcal{O}'$ if there exists $D \in \mathcal{O}$ and $D' \in \mathcal{O}'$ such that D and D' correspond.

Now put $r = s = n$, and assume that $p + q \le n$.

Lemma 4.1 *By the maps, we get*

$$\overline{\mathcal{O}_{p,q}} = \varphi_1(\varphi_2^{-1}(0)) \subset M_n \oplus M_n = \mathfrak{s},$$

i.e., the nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}_{p,q} \subset \mathfrak{s}$ corresponds to the trivial orbit $\{0\} \subset \mathfrak{s}' = M_{p,q} \oplus M_{q,p}$.

PROOF. Take $T \in \varphi_2^{-1}(0)$. Then we immediately have

$$\varphi_1(T) = \left[\begin{array}{c|c} & {}^t XZ \\ \hline {}^t WY & \end{array}\right] \in \mathfrak{X}_{\text{null}}$$

by the definition of φ_2. Moreover, it is easy to see that rank ${}^t XZ \le p$ and rank ${}^t WY \le q$. Thus we conclude that $\varphi_1(\varphi_2^{-1}(0)) \subset \overline{\mathcal{O}_{p,q}}$.

To see the reversed inclusion, let $X = Z = [1_p \; 0]$ and $W = Y = [0 \; 1_q]$. Then $\varphi_1(T) = x_{p,q}$ and $\varphi_2(T) = 0$. Thus $x_{p,q} \in \varphi_1(\varphi_2^{-1}(0))$. Since φ_1 and φ_2 are $K'_{\mathbb{C}} \times K_{\mathbb{C}}$-equivariant, we have $\mathcal{O}_{p,q} \subset \varphi_1(\varphi_2^{-1}(0))$. Taking the closure, we conclude that $\overline{\mathcal{O}_{p,q}} \subset \varphi_1(\varphi_2^{-1}(0))$. Q.E.D.

Next we consider the *null cone* defined by

$$\mathfrak{N} = \varphi_2^{-1}(0) = \left\{ \left[\begin{array}{c|c} X & Z \\ \hline Y & W \end{array}\right] \in \mathcal{W} \mid X\,{}^t Y = 0, W\,{}^t Z = 0 \right\}, \quad \mathcal{W} = M_{p+q,2n}.$$

It is $K'_{\mathbb{C}} \times K_{\mathbb{C}}$-stable.

Let $I = \mathbb{C}[\mathcal{W}]^{K_{\mathbb{C}}}$ be the algebra of $K_{\mathbb{C}}$-invariant polynomials on $\mathcal{W}$. By the classical invariant theory, the matrix elements of $X\,{}^t Y$ and $W\,{}^t Z$ generate I. Thus $\mathfrak{N}$ is the variety of zeros of all the invariants without constant term.

Proposition 4.2 *Assume that $p + q \le n$. Then the null cone $\mathfrak{N}$ is an irreducible variety of complete intersections. Its defining ideal is generated by $K_{\mathbb{C}}$-invariants of degree two. Moreover, if $p + q < n$, it is normal.*

PROOF. We apply [1, Theorem 2.2.11]. For details, see [8]. Q.E.D.

Let $\mathfrak{H}$ be the space of $K_{\mathbb{C}}$-harmonic polynomials, and J_+ the ideal generated by $K_{\mathbb{C}}$-invariants in $\mathbb{C}[\mathcal{W}]$ which vanish at the origin. The following lemma is well-known.

Lemma 4.3 *We have*

$$\mathbb{C}[\mathcal{W}] = \mathfrak{H} \oplus J_+.$$

Remark 4.4 If $p + q \le n$, then $\mathbb{C}[\mathcal{W}] \simeq \mathfrak{H} \otimes \mathbb{C}[\mathcal{W}]^{K_{\mathbb{C}}}$ holds.

Since $J_+ = \mathbf{I}(\mathfrak{N})$ is the defining ideal of $\mathfrak{N}$ by Proposition 4.2, the function ring on $\mathfrak{N}$ coincides with $\mathfrak{H}$. Note that $\mathfrak{H}$ carries the action of $K'_{\mathbb{C}} = GL_p \times GL_q$ and $K_{\mathbb{C}} = GL_n \times GL_n$. We record this as

Lemma 4.5 *The function ring on the null cone $\mathfrak{N} = \varphi_2^{-1}(0)$ coincides with the $K_{\mathbb{C}}$-harmonic polynomials on $\mathcal{W} = M_{p+q,2n}$:*

$$\mathbb{C}[\mathfrak{N}] \simeq \mathfrak{H} \quad (\textit{as a } K'_{\mathbb{C}} \times K_{\mathbb{C}}\textit{-module}).$$

Now we describe the $K'_{\mathbb{C}} \times K_{\mathbb{C}}$-module structure of $\mathfrak{H}$. Consider the two halves of the matrix T:

$$T^+ = \begin{bmatrix} X \\ Y \end{bmatrix} \in M_{p+q,n} =: \mathcal{W}^+, \quad T^- = \begin{bmatrix} Z \\ W \end{bmatrix} \in M_{p+q,n} =: \mathcal{W}^-.$$

Then by $U(p,q) \times U(n)$-duality, we know

$$\mathbb{C}[\mathcal{W}^+] \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} L(\tau_\lambda^{(n)})^* \boxtimes (\tau_\lambda^{(n)})^*,$$

as a $U(p,q) \times U(n)$-module. Note that $\mathbb{C}[\mathcal{W}^+]$ is the dual of the symmetric algebra $S(\mathcal{W}^+)$. Therefore, we take the dual on the right hand side. In this setting, the space of GL_n-harmonics of type $(\tau_\lambda^{(n)})^*$ has the form $\left((\tau_\alpha^{(p)})^* \otimes \tau_\gamma^{(q)}\right) \boxtimes (\tau_\lambda^{(n)})^*$, where $\left((\tau_\alpha^{(p)})^* \otimes \tau_\gamma^{(q)}\right)$ is the lowest $K' = U(p) \times U(q)$-type of $L(\tau_\lambda)^*$ tensored $\chi'_{n,n}$ ($\chi'_{n,n} = \det^{n/2} \boxtimes \det^{-n/2}$) and $\lambda = \alpha \odot \gamma$ (cf. (1.3)).

Let

$$\mathfrak{H}^+ = \mathfrak{H} \cap \mathbb{C}[\mathcal{W}^+], \qquad \mathfrak{H}^- = \mathfrak{H} \cap \mathbb{C}[\mathcal{W}^-].$$

Then we have

$$\mathfrak{H}^+ \simeq \sum_{\lambda \in \Lambda^+(p,q)}^{\oplus} \left((\tau_\alpha^{(p)})^* \otimes \tau_\gamma^{(q)}\right) \boxtimes (\tau_\lambda^{(n)})^*,$$

where $\lambda = \alpha \odot \gamma$ (cf. (1.3)). Similarly (but conjugated), we have

$$\mathbb{C}[\mathcal{W}^-] \simeq \sum_{\mu \in \Lambda^+(p,q)}^{\oplus} L(\tau_\mu^{(n)}) \boxtimes \tau_\mu^{(n)},$$

and

$$\mathfrak{H}^- \simeq \sum_{\mu \in \Lambda^+(p,q)}^{\oplus} \left(\tau_\beta^{(p)} \otimes (\tau_\delta^{(q)})^*\right) \boxtimes \tau_\mu^{(n)},$$

where $\mu = \beta \odot \delta$.

Theorem 4.6 *Put $\mathfrak{N} = \varphi_2^{-1}(0)$, the null cone defined by $K_{\mathbb{C}}$-invariants. Then we have*

$$\begin{aligned} \mathbb{C}[\mathfrak{N}] &\simeq \mathfrak{H} = \mathfrak{H}^+ \otimes \mathfrak{H}^- \\ &\simeq \sum_{\lambda,\mu \in \Lambda^+(p,q)}^{\oplus} \left(((\tau_\alpha^{(p)})^* \otimes \tau_\beta^{(p)}) \boxtimes (\tau_\gamma^{(q)} \otimes (\tau_\delta^{(q)})^*)\right) \boxtimes \left((\tau_\lambda^{(n)})^* \boxtimes \tau_\mu^{(n)}\right), \end{aligned}$$

as $K'_{\mathbb{C}} \times K_{\mathbb{C}}$-module. Furthermore the map φ_1^* induces a ring isomorphism:

$$\varphi_1^* : \mathbb{C}[\overline{\mathcal{O}_{p,q}}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}}. \tag{4.1}$$

PROOF. We have to show that (4.1) is an isomorphism.

Look at the original map $\varphi_1 : \mathcal{W} \longrightarrow \mathfrak{s}$. Then the induced map $\varphi_1^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[\mathcal{W}]$ is given by

$$\varphi_1^*(A_{ij})(T) = \sum_{k=1}^{p} X_{ki} Z_{kj}, \qquad \varphi_1^*(B_{ij})(T) = \sum_{k=1}^{q} W_{ki} Y_{kj},$$

where $T = \left[\begin{array}{c|c} X & Z \\ \hline Y & W \end{array}\right] \in \mathcal{W}$ and $\{(A_{ij}, B_{ij}) \mid 1 \leq i, j \leq n\}$ is the set of coordinates on $\mathfrak{s} = M_n \oplus M_n$. By the classical invariant theory, the right hand side of the above equations generate the ring of $K'_{\mathbb{C}}$-invariants in $\mathbb{C}[\mathcal{W}]$, so that $\operatorname{Image} \varphi_1^* = \mathbb{C}[\mathcal{W}]^{K'_{\mathbb{C}}}$.

Now consider the restriction $\varphi_1 : \mathfrak{N} \longrightarrow \mathfrak{s}$, which we denote by the same notation. By the above consideration, we know that the induced map actually gives an algebra morphism

$$\varphi_1^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[\mathfrak{N}]^{K'_{\mathbb{C}}} \subset \mathbb{C}[\mathfrak{N}] = \mathbb{C}[\mathcal{W}]/\mathbf{I}(\mathfrak{N}),$$

and it is surjective. Now, since $\mathbb{C}[\mathfrak{s}]/\operatorname{Ker} \varphi_1^* = \mathbb{C}[\overline{\mathcal{O}_{p,q}}]$ by Lemma 4.1, we conclude the result.

Q.E.D.

Corollary 4.7 *If $p + q < n$, the nilpotent orbit $\overline{\mathcal{O}_{p,q}}$ is an irreducible normal variety.*

Finally we are ready to prove the following proposition, which implies Theorem 2.4.

Proposition 4.8 *The associated variety of $\theta^{p,q}(\mathbf{1})$ is $\overline{\mathcal{O}_{q,p}}$.*

PROOF. We identify the nilpotent orbit $\mathcal{O}_{q,p}$ with a coadjoint orbit via Killing form. Then the corresponding orbit is $\mathcal{O}_{p,q}$. In this proof, we always consider $\mathcal{O}_{p,q}$ as a *coadjoint nilpotent orbit.* (See also the remark before Theorem 2.4.)

We take a coordinate system on $\mathfrak{s} = M_n \oplus M_n \ni (A, B)$ as $\{A_{ij}, B_{ij} \mid 1 \leq i, j \leq n\}$, as before. Note that we regard A_{ij} as a linear function on $\mathfrak{s}$ which extracts the (i, j)-element of A. So $A_{ij}, B_{ij} \in \mathbb{C}[\mathfrak{s}]$. Take one of the homogeneous defining equations $f(A, B) = 0$ of $\overline{\mathcal{O}_{p,q}}$. Then we have

$$\varphi_1^*(f(A, B)) = f(S, T); \quad S_{ij} = \varphi_1^*(A_{ij}) = \sum_{k=1}^{p} X_{ki} Z_{kj}, \; T_{ij} = \varphi_1^*(B_{ij}) = \sum_{l=1}^{q} W_{li} Y_{lj}.$$

By Theorem 4.6 and its proof, we know that $f(S, T) \in \mathbf{I}(\mathfrak{N})^{K'_{\mathbb{C}}}$. In particular, there exist polynomials $e_{kl}^{\pm} \in \mathbb{C}[\mathcal{W}]$ such that

$$f(S, T) = \sum_{\substack{1 \leq k \leq p \\ 1 \leq l \leq q}} e_{kl}^{+} \cdot \varphi_2^*(C_{kl}) + \sum_{\substack{1 \leq k \leq q \\ 1 \leq l \leq p}} e_{kl}^{-} \cdot \varphi_2^*(D_{kl}),$$

where $(C, D) \in M_{p,q} \oplus M_{q,p} = \mathfrak{s}'$ and $\{C_{kl}, D_{kl}\}$ is a system of coordinate functions on $\mathfrak{s}'$.

The Weil representation associated to the dual pair $U(p,q)\times U(n,n)$ is realized on $\mathbb{C}[\mathcal{W}]$ by the Fock model. Let $\widetilde{A}_{ij}$ be the operator of the Weil representation of the matrix

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline E_{ji} & 0 \end{array}\right] \in \mathfrak{s}.$$

Define $\widetilde{B}_{ij}$, $\widetilde{C}_{kl}$, $\widetilde{D}_{kl}$ similarly. Then we have

$$\begin{aligned} \widetilde{A}_{ij} &= 2\sum_{k=1}^{p} \frac{\partial^2}{\partial X_{ki}\partial Z_{kj}} - \frac{1}{2}\sum_{l=1}^{q} W_{li}Y_{lj}, \\ \widetilde{B}_{ji} &= -\frac{1}{2}\sum_{k=1}^{p} X_{ki}Z_{kj} + 2\sum_{l=1}^{q} \frac{\partial^2}{\partial W_{li}\partial Y_{lj}} \end{aligned}$$

up to non-zero constant multiplication.

If we take the K-stable good filtration of $\theta^{p,q}(\mathbf{1})$ induced from the total degree filtration of $\mathbb{C}[\mathcal{W}]$, $\widetilde{A}_{ij}$ and $\widetilde{B}_{ij}$ will act on the graded module $\operatorname{gr}\theta^{p,q}(\mathbf{1})$ as

$$\widetilde{B}_{ji} = -\frac{1}{2}\sum_{k=1}^{p} X_{ki}Z_{kj}, \quad \widetilde{A}_{ij} = -\frac{1}{2}\sum_{l=1}^{q} W_{li}Y_{lj}.$$

These formulas coincide with S_{ij} and T_{ij} above up to constant multiplication. Therefore we have

$$(-2)^{-\deg f} f(\widetilde{B}_{ji},\widetilde{A}_{ij}) = \sum_{1\le k\le p, 1\le l\le q} e^{+}_{kl}\cdot\varphi_2^*(C_{kl}) + \sum_{1\le k\le q, 1\le l\le p} e^{-}_{kl}\cdot\varphi_2^*(D_{kl})$$

on $\operatorname{gr}\theta^{p,q}(\mathbf{1})$. Note that

$$\varphi_2^*(C_{kl}) = \sum_{i=1}^{n} X_{ki}Y_{li}, \quad \varphi_2^*(D_{kl}) = \sum_{i=1}^{n} Z_{ki}W_{li}.$$

On the other hand, the operators $\widetilde{C}_{kl}$ and $\widetilde{D}_{kl}$ are given by

$$\begin{aligned} \widetilde{C}_{kl} &= 2\sum_{i=1}^{n} \frac{\partial^2}{\partial X_{ki}\partial Y_{li}} - \frac{1}{2}\sum_{j=1}^{n} W_{kj}Z_{lj} \equiv -\frac{1}{2}\sum_{j=1}^{n} W_{kj}Z_{lj}, \\ \widetilde{D}_{lk} &= -\frac{1}{2}\sum_{i=1}^{n} X_{ki}Y_{li} + 2\sum_{j=1}^{n} \frac{\partial^2}{\partial W_{lj}\partial Z_{kj}} \equiv -\frac{1}{2}\sum_{i=1}^{n} X_{ki}Y_{li}, \end{aligned}$$

which should vanish on $\operatorname{gr}\theta^{p,q}(\mathbf{1})$ because $\theta^{p,q}(\mathbf{1})$ is associated to the trivial representation of $U(p,q)^{\sim}$. Hence we conclude $\varphi_2^*(C_{kl})$ and $\varphi_2^*(D_{lk})$ act trivially on $\operatorname{gr}\theta^{p,q}(\mathbf{1})$. From this we see that $f(\widetilde{B}_{ji},\widetilde{A}_{ij}) = 0$ on $\operatorname{gr}\theta^{p,q}(\mathbf{1})$. This means that the associated variety of $\theta^{p,q}(\mathbf{1})$ is contained in $\overline{\mathcal{O}_{q,p}}$. Comparing dimension, we conclude that they must coincide. Q.E.D.

5 Generalities on Harish-Chandra modules

In this section, we consider general $(\mathfrak{g}, K)$-modules, where $\mathfrak{g}$ is the complexification of the Lie algebra of a semisimple Lie group G and K is a maximal compact subgroup of G.

Let H be a $(\mathfrak{g}, K)$-module which is K-admissible, i.e., $\dim \operatorname{Hom}_K(H, \tau) < \infty$ for any $\tau \in \operatorname{Irr}(K)$. We also assume that H is locally K-finite, which means that $H = H_K$ where H_K denotes the space of K-finite vectors in H. Then, $H^* = \operatorname{Hom}_{\mathbb{C}}(H, \mathbb{C})_K$ is also K-admissible and we have a canonical isomorphism $H \simeq (H^*)^*$. Note that H^* does *not* denote the algebraic dual of H.

Lemma 5.1 *Let H_1 be a $(\mathfrak{g}, K)$-module which is locally K-finite. Then, for any K-admissible $(\mathfrak{g}, K)$-module H_2 which is locally K-finite, we have*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(H_1 \otimes H_2^*, \mathbf{1}) \simeq \operatorname{Hom}_{(\mathfrak{g},K)}(H_1, H_2).$$

We shall omit the proof, as it is straightforward.

Corollary 5.2 *Let σ be a finite dimensional $(\mathfrak{g}, K)$-module. Then we have*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(H_1 \otimes H_2^*, \sigma) \simeq \operatorname{Hom}_{(\mathfrak{g},K)}(H_1, H_2 \otimes \sigma),$$

under the same assumption on H_1 and H_2 as in Lemma 5.1.

Proof. Since $H_1 \otimes H_2^*$ is locally K-finite, we can apply Lemma 5.1 and get

$$\operatorname{Hom}_{(\mathfrak{g},K)}(H_1 \otimes H_2^*, \sigma) \simeq \operatorname{Hom}_{(\mathfrak{g},K)}(H_1 \otimes H_2^* \otimes \sigma^*, \mathbf{1}).$$

Note that $H_2^* \otimes \sigma^* \simeq (H_2 \otimes \sigma)^*$ because σ is finite dimensional. Now apply Lemma 5.1 again. Q.E.D.

Corollary 5.3 *Let ω be the Harish-Chandra module of the Weil representation of $Mp(2n(p+q), \mathbb{R})$. We consider a dual pair $G \times G' = U(p, q) \times U(n)$ in $Mp(2n(p+q), \mathbb{R})$, and regard ω as a $(\mathfrak{g}, \widetilde{K})$-module. Then for any irreducible Harish-Chandra module L of $(\mathfrak{g}, \widetilde{K})$, we have*

$$\operatorname{Hom}_{(\mathfrak{g},\widetilde{K})}(\omega \otimes \omega^*, L^*) \simeq \operatorname{Hom}_{(\mathfrak{g},\widetilde{K})}(L \otimes \omega, \omega),$$

Proof. Note that $\omega \otimes \omega^*$ is locally $\widetilde{K}$-finite. Also we note that ω is $\widetilde{K}$-admissible. Now we can apply Lemma 5.1 twice and get the desired formula. Q.E.D.

6 Theta lift of a holomorphic discrete series

We keep to the notation in §2. In particular $L(\tau_\lambda)$ denotes the holomorphic discrete series representation of $U(p, q)^{\sim}$ which corresponds to $\tau_\lambda = \tau_\lambda^{(n)} \in \operatorname{Irr}(U(n))$. Note that $\lambda \in \Lambda^+(p, q)$, since $p + q \leq n$. We shall derive a K-type formula for its theta lift.

Lemma 6.1 *For any $\sigma \in \mathrm{Irr}(U(p,q)^{\sim})$, the theta lift $\theta^{p,q}(\sigma)$ is non-zero if and only if there exists a pair (λ, μ) such that*

$$L(\tau_\lambda) \otimes L(\tau_\mu)^*\big|_{\Delta U(p,q)^{\sim}} \longrightarrow \sigma$$

is a non-zero (hence surjective) quotient map. In this case, the maximal quotient $\Omega^{p,q}(\sigma)$ contains K-type $(\tau_\lambda \otimes \chi_{p,q}) \boxtimes (\tau_\mu \otimes \chi_{p,q})^*$ at least once.

Proof. Arguments are the same as in §2, though we use σ instead of $\mathbf{1}$. Q.E.D.

Now take $\sigma = L(\tau_\nu)$, a holomorphic discrete series. Then by Lemma 5.1, we have

$$\mathrm{Hom}_{(\mathfrak{g}', \widetilde{K}')}(L(\tau_\lambda) \otimes L(\tau_\mu)^*, L(\tau_\nu)) \simeq \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K}')}(L(\tau_\lambda) \otimes L(\tau_\mu)^* \otimes L(\tau_\nu)^*, \mathbf{1}_{U(p,q)^{\sim}}). \qquad (6.1)$$

Lemma 6.2 *Let $c^{\eta}_{\mu,\nu}$ be a branching coefficient defined by*

$$\tau_\mu \otimes \tau_\nu \simeq \sum_{\eta}^{\oplus} c^{\eta}_{\mu,\nu} \tau_\eta \quad (\text{as } GL_n\text{-modules}).$$

If n is even, we have

$$L(\tau_\mu)^* \otimes L(\tau_\nu)^* \simeq \sum_{\eta \in \Lambda^+(p,q)}^{\oplus} c^{\eta}_{\mu,\nu} L(\tau_{\eta - n/2\delta_{p,q}})^*;$$

$$\delta_{p,q} = \mathbb{I}_p \odot \mathbb{I}_q = (1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1).$$

Proof. Consider the Weil representation associated to the dual pair $U(p,q) \times U(2n)$. As a $(U(p,q) \times U(n))^2$-module, we have

$$\omega \otimes \omega \simeq \left(\sum_{\mu \in \Lambda^+(p,q)}^{\oplus} L(\tau_\mu^{(n)}) \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q}) \right) \otimes \left(\sum_{\nu \in \Lambda^+(p,q)}^{\oplus} L(\tau_\nu^{(n)}) \boxtimes (\tau_\nu^{(n)} \otimes \chi_{p,q}) \right)$$

$$\simeq \sum_{\mu,\nu}^{\oplus} \left(L(\tau_\mu^{(n)}) \otimes L(\tau_\nu^{(n)}) \right) \boxtimes \left((\tau_\mu^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\nu^{(n)} \otimes \chi_{p,q}) \right), \qquad (6.2)$$

while as a $U(p,q) \times U(2n)$-module

$$\omega \otimes \omega \simeq \sum_{\eta \in \Lambda^+(p,q)}^{\oplus} L(\tau_\eta^{(2n)}) \boxtimes (\tau_\eta^{(2n)} \otimes \chi_{p,q}). \qquad (6.3)$$

Since

$$\tau_\eta^{(2n)}\big|_{U(n) \times U(n)} \simeq \sum_{\mu,\nu}^{\oplus} c^{\eta}_{\mu,\nu} \tau_\mu^{(n)} \otimes \tau_\nu^{(n)},$$

we have

$$(6.3) = \sum_{\mu,\nu}^{\oplus} \left(\sum_{\eta}^{\oplus} c^{\eta}_{\mu,\nu} L(\tau_\eta^{(2n)}) \right) \boxtimes \left((\tau_\mu^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\nu^{(n)} \otimes \chi_{p,q}) \right). \qquad (6.4)$$

Comparing (6.2) and (6.4), we get

$$L(\tau^{(n)}_{\mu}) \otimes L(\tau^{(n)}_{\nu}) \simeq \sum_{\eta}{}^{\oplus} c^{\,\eta}_{\mu,\nu} L(\tau^{(2n)}_{\eta}). \tag{6.5}$$

Since $p+q$ is smaller than n and hence $2n$,

$$L(\tau^{(2n)}_{\eta}) \simeq L(\tau^{(n)}_{\eta-n/2\delta_{p,q}}), \quad \delta_{p,q} = \mathbb{I}_p \odot \mathbb{I}_q$$

holds if we shift η by $n/2\delta_{p,q}$. Here we use the assumption that n is even. Also we identify $\eta = (\alpha, 0, \dots, 0, \gamma^*)$ ($\alpha \in \mathcal{P}_p, \gamma \in \mathcal{P}_q$) of length $2n$ (zero occurs $2n-(p+q)$-times between α and γ^*) and $\eta = (\alpha, 0, \dots, 0, \gamma^*)$ (zero occurs $n-(p+q)$-times between α and γ^*) of length n.

Take the K-finite part of the dual of (6.5). Since the K-types of $L(\tau^{(n)}_{\mu}) \otimes L(\tau^{(n)}_{\nu})$ has finite multiplicities, we get the formula in the lemma. (Or, we can consider $\omega^* \otimes \omega^*$ from the beginning.) Q.E.D.

We apply the above lemma to (6.1). Then we get

$$\operatorname{Hom}_{(\mathfrak{g}', \widetilde{K}')}(L(\tau_\lambda) \otimes L(\tau_\mu)^*, L(\tau_\nu)) \simeq \prod_{\eta} c^{\,\eta}_{\mu,\nu} \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K}')}(L(\tau_\lambda) \otimes L(\tau_{\eta-n/2\delta_{p,q}})^*, \mathbf{1}_{U(p,q)^{\sim}}).$$

The last expression is already studied by Lee-Zhu (see Lemma 2.1 and Corollary 2.2), which says that

$$\dim \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K}')}(L(\tau_\lambda) \otimes L(\tau_{\eta-n/2\delta_{p,q}})^*, \mathbf{1}_{U(p,q)^{\sim}}) \leq 1,$$

and the above mentioned dimension is one if and only if $\lambda = \eta - n/2\delta_{p,q}$ holds. So we get

Theorem 6.3 *Assume that n is even. Let $L(\tau_\nu)$ be a holomorphic discrete series and consider the theta lift $\theta^{p,q}(L(\tau_\nu)) \in \operatorname{Irr}(U(n,n)\tilde{})$. Then we have*

$$\theta^{p,q}(L(\tau_\nu))\big|_{\widetilde{K}} \simeq \sum_{\eta,\mu}{}^{\oplus} c^{\,\eta}_{\mu,\nu} (\tau^{(n)}_{\eta-n/2\delta_{p,q}} \otimes \chi_{p,q}) \boxtimes (\tau^{(n)}_{\mu} \otimes \chi_{p,q})^*.$$

PROOF. Put $\sigma = L(\tau_\nu)$. By the arguments around [7, Lemma 2.2], we know that $\Omega^{p,q}(\sigma)$ is already irreducible for a (holomorphic) discrete series representation σ. Note that $p+q \leq n$. By the calculation above, we know that $\Omega(\sigma)\big|_{\widetilde{K}}$ is exactly equal to the right hand side of the desired formula. Q.E.D.

We shall omit the result for odd n, which is similar.

Problem 6.4 Consider the same thing for a finite dimensional irreducible representation $\sigma \in \operatorname{Irr}(U(p,q)\tilde{})$, *which is not unitary.*

7 Further results

Here we would like to indicate several generalizations of the results of this note.

- We can consider the associated cycles of the theta lift of holomorphic discrete series representations. In particular, if we take a spherical holomorphic discrete series, which means that the minimal K-type is one-dimensional, then it has a multiplicity-free associated cycle with an irreducible associated variety.
- We have a generalization of the results here to the pair $U(p,q) \times U(r,s)$ in the stable range. However, the description of the parameters gets more complicated.
- We have similar results for the other type I dual pairs in stable range (cf. [8]).

These generalizations will be treated in a forthcoming paper.

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