Representations of reductive groups and invariant theory

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1 Introduction

2 First Fundamental Theorem (FFT)

3 Second Fundamental Theorem (SFT)

4 Geometric invariant theory (a first step)
What is invariants?
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Examples of invariants in broader sense

- numbers ... of finite sets
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- **numbers** ... of finite sets
- **dim** ... of finite vector spaces
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- **genus** ... of compact 2-dimensional surfaces
  or we should say...
  **Euler characteristic** ... of manifolds
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**Classification problem**

⇒ study of equivalence classes
⇒ invariants
There are more sophisticated invariants...
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- Alexander/Jones/Homfly/Kauffman polynomials ... of knots and links
- Vasiliev invariants
- Chern-Simons invariants
- Now there are so many invariants, quantum invariants, ...

- Donaldson invariants
- Seiberg-Witten invariants
- Gromov-Witten invariants ... quantum cohomology

- Iwasawa invariants ... for class field theory
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   Or more generally, quadratic form of signature \((p, q)\) \((p + q = n)\)
   \[ x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \]
   \[ \implies \text{Sylvester’s law of inertia} \]
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2. determinant: \( \det X \)
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   trace : \(\text{trace } X\)

   \[\text{trace}(gXg^{-1}) = \text{trace } X\quad g: \text{invertible matrix}\]
The discriminant: $\Delta(f)$ is a $\text{SL}_2$-invariant.

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

$$= a_0 \prod_{j=1}^{n} (x - \zeta_j)$$

$$\Delta(f) := a_0^{2n-2} \prod_{i<j}(\zeta_i - \zeta_j)^2 \quad \text{... polynomial in } a = (a_0, a_1, \ldots, a_n)$$
discriminant : \( \Delta(f) \) \( \cdots \) SL\(_2\)-invariant

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resultant : \( R(f, g) \) \( \cdots \) SL\(_2\)-invariant

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f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = a_0 \prod_{i=1}^{n} (x - \zeta_i)
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\[
g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m = b_0 \prod_{j=1}^{m} (x - \xi_j)
\]

\[
R(f, g) := a_0^m b_0^n \prod_{i,j} (\zeta_i - \zeta_j) \quad \cdots \text{polynomial in } a \& b
\]
These are all \textit{polynomial invariants} and
These are all polynomial invariants and related to some group action
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Here is a summary:

<table>
<thead>
<tr>
<th>distance</th>
<th>$O_n \hookrightarrow \mathbb{R}^n :$ orthogonal group</th>
</tr>
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<tbody>
<tr>
<td>quadratic form</td>
<td>$O_{p,q} \hookrightarrow \mathbb{R}^n :$ indefinite orth group</td>
</tr>
<tr>
<td>det $X$, trace $X$</td>
<td>$GL_n \hookrightarrow M_n :$ adjoint action</td>
</tr>
<tr>
<td>$\Delta(f), R(f, g)$</td>
<td>$SL_2 \hookrightarrow \mathbb{C}[x, y]_n$</td>
</tr>
</tbody>
</table>

Here $GL_n = \{g : n \times n$-matrix $| \exists g^{-1} \iff \det g \neq 0\}$

$O_{p,q} = \{g \in GL_n \mid \|g x\|_{p,q} = \|x\|_{p,q}\}$  \hspace{0.5cm} (n = p + q)

where $\|x\|_{p,q} = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$

$SL_n = \{g \in GL_n \mid \det g = 1\}$ \hspace{0.5cm} example of reductive groups
Abstract setting

$G : \text{(linear) group } \curvearrowright X \subset \mathbb{C}^N : \text{algebraic action of } G \text{ on } X$
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\[ G : \text{linear group} \bowtie X \subset \mathbb{C}^N : \text{algebraic action of } G \text{ on } X \]

**Definition (algebraic action)**

\[ \rho : G \times X \rightarrow X : \text{polynomial function s.t.} \]

1. \( \rho(e, x) = x \)
2. \( \rho(gh, x) = \rho(g, \rho(h, x)) \)

Notation: \( g \cdot x = gx = \rho(g, x) \)
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Polynomial functions and invariants:

$\mathbb{C}[X] := \{f : X \to \mathbb{C} \mid f \text{ is polynomial function} \} : \text{ring of regular functions}$

$\mathbb{C}[X]^G := \{f \in \mathbb{C}[X] \mid f(g^{-1} \cdot x) = f(x) \ (\forall g \in G)\} : \text{ring of invariants}$
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functions which are constant along orbits
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$\implies \mathbb{C}[X]^G \text{ is graded by degree of polynomials, i.e., graded algebra}$
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  - **Fourier transform**, etc.
Fundamental problems of invariant theory

Two classical problems ...
Fundamental problems of invariant theory

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1. Find the ring generators $\{\Delta_i\}_{i \in I} \subset \mathbb{C}[X]^G$
   
   Question:  $\exists$ finite number of generators?
   Can choose a good basis?

   FFT = First Fundamental Theorem
Fundamental problems of invariant theory

Two classical problems ...

1. Find the ring generators $\{\Delta_i\}_{i \in I} \subset \mathbb{C}[X]^G$
   
   Question: Does there exist a finite number of generators?
   Can we choose a good basis?

   $\text{FFT} = \text{First Fundamental Theorem}$

2. Find all the relations among $\{\Delta_i\}_{i \in I}$
   
   Question: What is the transcending degree?
   What are the singularities?

   $\text{SFT} = \text{Second Fundamental Theorem}$
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There are many kinds of answers \( \cdots \)
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Final Goal

Better understanding of $\mathbb{C}[X]^G$ in \textit{geometric terms}.
Understanding of the original action $G \curvearrowright X$ through it.
Basic setting

$G$: reductive, linear algebraic group
Basic setting

\( G : \text{ reductive, linear algebraic group} \)

**Definition (algebraic group)**

algebraic group = Zariski closed subgroup in \( \text{GL}_N(\mathbb{C}) \)

Zariski closed = solutions of the system of polynomial equations
Basic setting

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Definition (algebraic group)

algebraic group = Zariski closed subgroup in $\text{GL}_N(\mathbb{C})$

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Definition (reductive)

reductive group = nilpotent radical is trivial

(if /$k$, $k$ being algebraically closed, $\text{char } k = 0$)

= $\forall$ finite dim representation is completely reducible

= $\forall$ finite dim repr is decomposed into the direct sum of irreducibles

$V$ : reducible $\iff V = U_1 \oplus U_2$ ($\exists U_i$ : subrepresentation)

irreducibles = basis unit (atom) of representation
Reductive groups

Example (reductive groups)

\[ \mathbb{T} = (\mathbb{C}^\times)^m : \text{torus} \]

\[ \text{GL}_n, \text{SL}_n, \text{O}_n, \text{SO}_n = \text{O}_n \cap \text{SL}_n, \text{Sp}_{2n} : \text{classical groups} \]

\[ G_2, F_4, E_6, E_7, E_8 : \text{exceptional groups} \]
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1. *Product of reductive groups is reductive*
2. \( G : \text{reductive}, H \subset G : \text{normal} \implies G/H \text{ reductive (quotient)} \)
3. \( G^\circ : \text{reductive} \implies G : \text{reductive} \quad (G^\circ : \text{identity component}) \)

*Extension by finite group (\( \#G/G^\circ < \infty \))
Finite generation of invariants

Here is one of the best answer to FFT

**Theorem (D. Hilbert 1990, 1993)**

\[ G : \text{reductive} \quad \sim \quad V = \mathbb{C}^n : \text{vector space (linear repr)} \]

\[ \implies \quad \mathbb{C}[V]^G : \text{finitely generated algebra} / \mathbb{C} \]
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Remark

\exists \text{ counter example for non-reductive } G

\[ \ldots \text{ Nagata (1959) : Hilbert’s 14th problem} \]
Recent work by Mukai (2005) \ldots Rich examples of finite generation even when \( G \) is not reductive
Example of actions of finite groups

\[ \#G < \infty \implies G : \text{reductive} \]

\[ \mathbb{C}[V]^G = \{ R(f) \mid R(f)(x) = \frac{1}{\#G} \sum_{g \in G} f(g^{-1}x) \} \]

\( R(f) \) : Reynolds operator (projection to invariants)
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**Theorem**

\( G : \text{a finite reflection group} \)

\( \{ \Delta_1, \ldots, \Delta_l \} \subset \mathbb{C}[V]^G : \text{minimal homogeneous generators} \)

\[ \implies \{ d_k = \deg \Delta_k \mid 1 \leq k \leq l \} : \text{uniquely determined (exponents) } \]
Rational invariants and Galois theory

Theorem

\[ G : \text{finite group} \implies \mathbb{C}(V)^G = Q(\mathbb{C}[V]^G) : \text{quotient field} \quad \& \]
\[ [\mathbb{C}(V) : \mathbb{C}(V)^G] = \#G \]
Rational invariants and Galois theory

**Theorem**

\[ G : \text{finite group} \implies \mathbb{C}(V)^G = Q(\mathbb{C}[V]^G) : \text{quotient field} \quad \&\quad [\mathbb{C}(V) : \mathbb{C}(V)^G] = \#G \]

\[ \mathbb{C}(V) : \text{Galois extension} \text{ of } \mathbb{C}(V)^G \text{ with Galois group } G \]

i.e.,

**Study of** \( \mathbb{C}[V]^G \leftrightarrow \text{Galois theory for rings} \)
Example: symmetric group action

\[ G = \mathfrak{S}_n \curvearrowright V = \mathbb{C}^n : \text{action by coordinate change} \]
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Generators of the ring of invariants \( \mathbb{C}[V]^G \): 

- \{elementary symm fun \( e_k(1 \leq k \leq n) \}\}

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\prod_{i=1}^{n} (1 + tx_i) = \sum_{k=0}^{n} e_k(x) t^k
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- \{power sum \( p_k(1 \leq k \leq n)\}\} \quad p_k(x) = \sum_{i=1}^{n} x_i^k

- \{complete symm fun \( h_k(1 \leq k \leq n)\}\} \quad \prod_{i=1}^{n} (1 - tx_i)^{-1} = \sum_{k=0}^{\infty} h_k(x) t^k
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\[ \mathbb{C}[V]^G = \{\text{symmetric polynomials}\} : \text{invariants} \]

Generators of the ring of invariants \( \mathbb{C}[V]^G \) :

- \{\text{elementary symm fun } e_k(1 \leq k \leq n)\} \quad \prod_{i=1}^{n} (1 + tx_i) = \sum_{k=0}^{n} e_k(x)t^k

- \{\text{power sum } p_k(1 \leq k \leq n)\} \quad p_k(x) = \sum_{i=1}^{n} x_i^k

- \{\text{complete symm fun } h_k(1 \leq k \leq n)\} \quad \prod_{i=1}^{n} (1 - tx_i)^{-1} = \sum_{k=0}^{\infty} h_k(x)t^k

Exponents \( \{1, 2, \ldots, n\} \)

Generators are algebraically independent
Example: symmetric group action (continued)

Define a quotient map $\Phi : V \to \mathbb{C}^n$ by $\Phi(\nu) = (e_1(\nu), e_2(\nu), \ldots, e_n(\nu))$
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**Lemma**

$\Phi : V \to \mathbb{C}^n$ is surjective & $\mathfrak{S}_n$-invariant

Every fiber $\Phi^{-1}(y)$ ($y \in \mathbb{C}^n$) is a single $\mathfrak{S}_n$-orbit
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$\Phi$ surjective $\iff$ the fundamental theorem of algebra (Gauss’s theorem)

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Thus we conclude $V/\mathfrak{S}_n \simeq \mathbb{C}^n$ via the quotient map $\Phi$

$\Phi: V \to \mathbb{C}^n/\mathfrak{S}_n = \mathbb{C}^n$ : generically $[\mathfrak{S}_n : 1]$ map (Galois covering)

Generic fiber $\simeq \mathfrak{S}_n$, inherits regular representation of $\mathfrak{S}_n$
Orthogonal invariants

\[ G = O_n \leftarrow V = \mathbb{C}^n : \text{vector representation (mult of matrix against vector)} \]

Problem

*Describe the invariants for* \( G \leftarrow V \oplus \cdots \oplus V = V^{\oplus m} \)

\[ U := \mathbb{C}^m \quad \implies \quad V^{\oplus m} \cong V \otimes U \cong M_{n,m} \quad \text{coordinates } x_{ij} \text{ on } M_{n,m} \]
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FFT for orthogonal invariants:

Theorem (H. Weyl 1939)

\[ \mathbb{C}[V^{\oplus m}]^{O_n} = \mathbb{C}[z_{ij} \mid 1 \leq i \leq j \leq m] \quad \text{where} \; z_{ij} = \sum_{k=1}^{n} x_{ki}x_{kj} \]

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**Example**

\[ m = 1 : \quad \mathbb{C}[V]^{O_n} = \mathbb{C}[\xi] \quad \xi = x_1^2 + \cdots + x_n^2 \]
Contraction invariants

\[ G = \text{GL}_n \leftarrow V = \mathbb{C}^n \implies \mathbb{C}[V \oplus m]^G = \mathbb{C} : \text{trivial (NO invariants)} \]
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**Problem**

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\[ z_{ij} = \sum_{k=1}^n x_{ki} y_{kj} : \text{contraction of } X \text{ and } Y \]

\[ X = (x_{ij}) \in M_{n,p}, \ Y = (y_{ij}) \in M_{n,q} \implies Z = (z_{ij}) = {}^tXY \in M_{p,q} \]
Second Fundamental Theorem $= \text{SFT}$

describing relations among generators ...
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**Theorem (Shephard-Todd 1954)**

Assume $\#G < \infty$, $G \curvearrowright V$ : linear representation
$\mathbb{C}[V]^G$ is a polynomial ring (no relations)
\[ \iff G \text{ is a pseudo-reflection group} \]

**Remark**

$s$: pseudo-reflection $= \exists U \subset V$ : $(n - 1)$-dim s.t. $s|_U = \text{id}_U$
pseudo-reflection group $= \text{finite group generated by pseudo-reflections}$
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pseudo-reflection group = finite group generated by pseudo-reflections

If \( \exists \) relations, what we can do? Namely

**Problem**

*How to describe relations among generators?*
Return to the general situation $G \acts X$ ( $X \subset \mathbb{C}^N$ )

$G$: reductive; $X$: affine variety (solutions of polynomial equations)

$\{\Delta_1, \ldots, \Delta_m\} \subset \mathbb{C}[X]^G$ : generators of invariants
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Algebra morphism:

$$\Phi^* : \mathbb{C}[y_1, \ldots, y_m] \ni F(y) \mapsto F(\Delta_1, \ldots, \Delta_m) \in \mathbb{C}[X]^G$$

$\exists$ relation $F(\Delta_1, \ldots, \Delta_m) \equiv 0 \iff F(y) \in \text{Ker } \Phi^* =: I$

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**Theorem (Hilbert’s basis theorem)**

\forall ideal $I \subset \mathbb{C}[y]$ admits finite $\#$ of generators $\{F_1, \ldots, F_\ell\}$

**Notation**

$$
I = (F_1, \ldots, F_\ell) = \sum_{j=1}^{\ell} \mathbb{C}[y]F_j : \text{ideal generated by } \{F_1, \ldots, F_\ell\}
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SFT describes the generators of relations $\{F_1, \ldots, F_\ell\}$ completely, which are satisfied by invariants $\{\Delta_1, \ldots, \Delta_m\}$.
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**Example (orthogonal invariants)**

Recall \( G = O_n \subseteq V^\oplus m \cong M_{n,m} \) \((V = \mathbb{C}^n)\)
coordinates \( x_{ij} \) on \( M_{n,m} \) \(\implies\) orthog invariants: \( z_{ij} = \sum_{k=1}^n x_{ki} x_{kj} \)
\( X = (x_{ij}) \in M_{n,m} \implies Z = (z_{ij}) = tXX \in \text{Sym}_m \)

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Recall $G = O_n \subset V^{\oplus m} \cong M_{n,m}$ ($V = \mathbb{C}^n$) coordinates $x_{ij}$ on $M_{n,m} \implies$ orthog invariants: $z_{ij} = \sum_{k=1}^n x_{ki}x_{kj}$

$x = (x_{ij}) \in M_{n,m} \implies Z = (z_{ij}) = tXX \in \text{Sym}_m$

1. $m \leq n \implies \{z_{ij}\}$ : alg independent (no relation)
2. $m > n \implies Z = (z_{ij})$ is of rank $n$
   (i.e., relations are $(n + 1)$-th minors in $\text{Sym}_m$)
Geometric point of view

\[ l \subset \mathbb{C}[y] : \text{ideal of relations (prime)} \]
\[ \iff Y = \{ y \in \mathbb{C}^m \mid F(y) = 0 \ (\forall F \in l) \} \subset \mathbb{C}^m : \text{variety} \]
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(reduced ideals in \( \mathbb{C}[y] \)) \( \ni I \overset{\text{biject}}{\leftrightarrow} \)
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- \( Y = \mathbb{V}(I) = \{ y \in \mathbb{C}^m \mid F(y) = 0 \ (\forall F \in I) \} \subset \mathbb{C}^m \)

\( I : \text{prime ideal} \iff Y : \text{irreducible} \)

(i.e., \( Y = Y_1 \cup Y_2 \ (Y_i : \text{Z closed}) \implies Y = Y_1 \text{ or } Y = Y_2 \))
Conclusion:

\[ \text{SFT = describe algebraic variety defined by relations among invariants} \]

To be continued...