BERNSTEIN DEGREE AND ASSOCIATED CYCLES OF HARISH-CHANDRA MODULES – HERMITIAN SYMMETRIC CASE –

by

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Dedicated to Professor Ryoshi Hotta on his 60th anniversary

Abstract. — Let $\tilde{G}$ be the metaplectic double cover of $Sp(2n,\mathbb{R})$, $U(p,q)$ or $O^*(2p)$. we study the Bernstein degrees and the associated cycles of the irreducible unitary highest weight representations of $\tilde{G}$, by using the theta correspondence of dual pairs. The first part of this article is a summary of fundamental properties and known results of the Bernstein degrees and the associated cycles. Our first result is a comparison theorem between the K-module structures of the following two spaces; one is the theta lift of the trivial representation and the other is the ring of regular functions on its associated variety. Secondarily, we obtain the explicit values of the degrees of some small nilpotent $K_{\mathbb{C}}$-orbits by means of representation theory. The main result of this article is the determination of the asssociated cycles of singular unitary highest weight representations, which are the theta lifts of irreducible representations of certain compact groups. In the proofs of these results, the multiplicity free property of spherical subgroups and the stability of the branching coefficients play important roles.

Résumé. — Soit $\tilde{G}$ la double revêtement métaplectique de $Sp(2n,\mathbb{R})$, $U(p,q)$ ou $O^*(2p)$. Nous étudions les degré de Bernstein et les cycles associés des représentations irréductibles unitaires de $\tilde{G}$ avec plus haut poids, en utilisant la thêta correspondance par paires duales. La première partie de cet article est un abrégé de propriétés fondamentaux et résultats connus des degrés de Bernstein et les cycles associés. Notre premier résultat est un théorème comparatif entre les structures en tant que K-modules de deux espaces suivants; l'un est le thêta relèvement de la représentation évidente, l'autre est l'anneau de fonctions régulières sur sa variété associée. Deuxièmement, nous obtenons les valeurs concrètes des degrés de quelques petites $K_{\mathbb{C}}$-orbites nilpotentes au moyen de théorie de la représentation. Le principal résultat de cet article est la détermination des cyles associés de représentations singuliéres unitaires avec plus haut poids, qui sont les thêta relèvements des représentations irréductibles des certains groupes compacts. Dans les démonstrations de ces résultats, la propriété sans multiplicité de sou-groupes sphériques et la stabilité des coefficients de brnchement jouent rôles importants.

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Contents

Introduction

Let G be a semisimple (or more generally, reductive) Lie group. For an irreducible admissible representation π of G, there exist several important invariants such as irreducible characters, primitive ideals, associated varieties, asymptotic supports, Bernstein degrees, Gelfand-Kirillov dimensions, etc. They are interrelated with each other, and intimately related to the geometry of coadjoint orbits.

For example, at least if G is compact and π is finite dimensional, the character of π is the Fourier transform of an orbital integral on a semisimple coadjoint orbit ([**29**]). This is also the case for a general semisimple G and fairly large family of the representations (see [**41**]). This intimate relation between coadjoint orbits and irreducible representations invokes the philosophy of so-called *orbit method*, which is exploited by pioneer works of Kirillov and Kostant, and is now being developed by many contributors. However, for a general semisimple Lie group G, it seems that the orbit method still requires much to do. In particular, we should understand some small representations corresponding to nilpotent coadjoint orbits, which are called *unipotent*.

On the other hand, by definition, most of invariants are directly related to *nilpotent* coadjoint orbits. In a sense, the corresponding nilpotent orbits represent the leading term of irreducible characters ([**1**], [**44**]). The invariants of large representations correspond to the largest nilpotent coadjoint orbit, namely, the principal nilpotent orbit. For large representations, the orbit method seems to behave considerably well. Therefore we are now interested in 'small' representations whose invariants are related to smaller nilpotent coadjoint orbits.

One extreme case is the case of finite dimensional representations. In this case, however, the corresponding orbit is zero, and there is not a so much interesting phenomenon. The next to the extreme case is the case of minimal representations, which corresponds to the minimal nilpotent orbit. The minimal nilpotent orbit is unique in the sense that it is the only orbit among non-zero nilpotent ones with the smallest possible dimension. These representations have a simple structure. For example, their K-type structure is in a ladder form and is multiplicity free ([**50**]). Against its simple structure, though, systematic and thorough study of the minimal representations is still progressing now through the works of Kostant-Brylinski and many other mathematicians. If we turn our attention to the small representations other than minimal ones, it seems that there is relatively less knowledge on them up to now. In this paper, we study small representations which are unitary lowest (or highest) weight representations of G. Such representations exist if and only if G/K enjoys a structure of Hermitian symmetric space, where K denotes a maximal compact subgroup of G.

To be more specific, let us introduce notations. We assume that the symmetric space G/K is irreducible and Hermitian. Moreover, we assume that G is classical other than $SO(n, 2)$, i.e., $G = Sp(2n, \mathbb{R}), U(p, q)$ or $O^*(2p)$. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the Cartan decomposition with respect to K. We denote the complexified decomposition by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Since G/K is an irreducible Hermitian symmetric space, the induced adjoint representation of K on $\mathfrak{p}$ breaks up into precisely two irreducible components $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Note that, as a representation of K, $\mathfrak{p}^-$ is contragredient to $\mathfrak{p}^+$ via the Killing form. We extend this representation of K to the representation of the complexification $K_{\mathbb{C}}$ of K holomorphically.

Let L be an irreducible unitary lowest weight module of G. Then it is well-known that the associated variety of L, denoted by $\mathcal{AV}(L)$, is the closure of a single nilpotent $K_{\mathbb{C}}$-orbit contained in $\mathfrak{p}^-$ (we choose an appropriate positive system which is compatible with $\mathfrak{p}^+$).

Put $r = \mathbb{R}\text{-rank}\, G$, the real rank of G. Then there exist exactly $(r+1)$ nilpotent $K_{\mathbb{C}}$-orbits $\{\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_r\}$ in $\mathfrak{p}^-$. We choose an indexing of the orbits so that $\dim \mathcal{O}_{i-1} < \dim \mathcal{O}_i$ holds for $1 \le i \le r$; in particular, $\mathcal{O}_0 = \{0\}$ is the trivial one, and $\mathcal{O}_r$ is the open dense orbit. Most of lowest weight representations L correspond to the largest orbit $\mathcal{O}_r$. For example, the associated variety of a holomorphic discrete series (or its limit) is $\overline{\mathcal{O}_r} = \mathfrak{p}^-$. The invariants of the holomorphic discrete series representations are completely understood (see [**14**], [**43**], [**7**]; also see §2.4 below). However, for each orbit $\mathcal{O}_m$ $(0 < m < r)$, there exists a relatively small family of lowest weight representations whose associated variety is indeed the closure of the orbit $\mathcal{O}_m$. Thanks to the theory of reductive dual pairs via the Weil representation of metaplectic groups, we have a complete knowledge of such a family of lowest weight representations (at least for classical groups listed above).

Although we can define a specific 'small' representations even for the largest orbit $\mathcal{O}_r$, we restrict ourselves to the case $\mathcal{O}_m$ $(m < r)$ in this introduction. Then there exists a compact group G_2 corresponding to each m (cf. §3, Table 2) such that (G_1, G_2) forms a dual pair in a large symplectic group $Sp(2N, \mathbb{R})$. Let $Mp(2N, \mathbb{R})$ be the metaplectic double cover of $Sp(2N, \mathbb{R})$. We denote by $\widetilde{H} \subset Mp(2N, \mathbb{R})$ the inverse image of a subgroup $H \subset Sp(2N, \mathbb{R})$ of the covering map.

The family of unitary irreducible lowest weight representations of $\widetilde{G}$ whose associated variety is $\overline{\mathcal{O}_m}$ is parametrized by $\mathrm{Irr}(G_2)$, the set of the irreducible finite dimensional representations of G_2. We denote the lowest weight representation of $\widetilde{G}$

corresponding to $\sigma \in \mathrm{Irr}(G_2)$ by $L(\sigma)$ (see §5 for precise description). Roughly, the correspondence $\sigma \mapsto L(\sigma)$ is the theta lift after twisted by a certain unitary character of $\widetilde{G_2}$.

Our first observation is the following.

Theorem A. — *Let $\mathbf{1}_{G_2}$ be the trivial representation of G_2 and $L(\mathbf{1}_{G_2})$ the unitary lowest weight representation of $\widetilde{G}$ corresponding to $\mathbf{1}_{G_2}$. The Bernstein degree of $L(\mathbf{1}_{G_2})$ coincides with the degree of the closure of the nilpotent orbit $\overline{\mathcal{O}_m}$ (defined in the sense of algebraic geometry) ;*

$$\mathrm{Deg}\, L(\mathbf{1}_{G_2}) = \deg \overline{\mathcal{O}_m}.$$

We also get an explicit and computable formula for $\mathrm{Deg}\, L(\mathbf{1}_{G_2})$.

Note that the varieties $\overline{\mathcal{O}_m}$ are determinantal varieties of various type and an explicit formula of their degree is known as *Giambelli-Thom-Porteous formula*. Our representation theoretic proof of the formula seems new, and gives an alternative proof.

To prove Theorem A, we construct a $K_{\mathbb{C}}$-equivariant map $\psi : V \to \overline{\mathcal{O}_m}$, where V is a certain $K_{\mathbb{C}} \times (G_2)_{\mathbb{C}}$-module. This map induces an algebra isomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \xrightarrow{\sim} \mathbb{C}[V^*]^{(G_2)_{\mathbb{C}}},$$

which means that $\overline{\mathcal{O}_m} = V//(G_2)_{\mathbb{C}}$. The map ψ is closely related to the dual pair (G, G_2), and we call it *unfolding* of $\overline{\mathcal{O}_m}$. By this, the proof of Theorem A reduces to a problem of classical invariant theory.

The 'smallest' unipotent representation attached to the orbit $\mathcal{O}_m$ should be realized on the section of a certain line bundle on $\mathcal{O}_m$ called half-form bundle ([**5**], [**6**], [**52**]). We investigate such half-form bundles, and get an evidence of strong relationship between the space of global sections of the half-form bundles and $L(\sigma)$, where σ is a special one-dimensional character of G_2.

Next, let us consider a general unitary lowest weight module $L(\sigma)$ ($\sigma \in \mathrm{Irr}(G_2)$). We describe its K-type decomposition and the Poincaré series in terms of certain branching coefficient of finite dimensional representations of general linear groups and G_2. Such descriptions are well-known among experts. However, references to them are scattered in many places, and sometimes their treatments are ad hoc. Since we need an explicit and unified picture for the K-types of $L(\sigma)$, we reproduce the decompositions in the sequel.

Now our main theorem says

Theorem B. — *Let $L(\sigma)$ be an irreducible unitary lowest weight module of $\widetilde{G}$ corresponding to $\sigma \in \mathrm{Irr}(G_2)$. Then its Bernstein degree is given by*

$$\mathrm{Deg}\, L(\sigma) = \dim \sigma \cdot \deg \overline{\mathcal{O}_m}.$$

There is a notion of *associated cycle* which is a refinement of the notion of associated variety. Roughly speaking, it expresses associated variety with multiplicity. For a precise definition, see §§1.1 and 1.3. Then the following is an immediate corollary to Theorem B.

Theorem C. — *The associated cycle of $L(\sigma)$ is given by $\mathcal{AC}\,(L(\sigma)) = \dim\sigma \cdot [\overline{\mathcal{O}_m}]$.*

The proof of Theorem B is based on the theory of multiplicity free action of algebraic groups, which is a subject of §8. The key ingredients of the proof are multiplicity free property of spherical subgroups and Sato's summation formula of the stable branching coefficients.

Lastly, we would like to comment on several aspects of our results.

First, the Bernstein degree of an irreducible representation π is closely related to the dimension of its "Whittaker vectors". In fact, for large representations, Matumoto proved that the Bernstein degree and the dimension of algebraic Whittaker vectors coincide ([**36**]). For 'small' representations, we cannot hope the same story, because they do not have any Whittaker vector in a naive sense. However, for complex semisimple Lie groups, Matumoto observed that the finite-dimensionality and non-vanishing of the space of certain degenerate Whittaker vectors determines the wave front set of π ([**34**], [**35**]). Recently, Yamashita has found a strong relation between the multiplicity of associated cycles and the dimension of generalized Whittaker vectors in the case of unitary highest weight module ([**54**]).

Second, let us consider the (twisted) theta correspondence (or Howe correspondence, dual pair correspondence, ...) between $L(\sigma) \in \mathrm{Irr}(\widetilde{G})$ and $\sigma \in \mathrm{Irr}(G_2)$. Since G_2 is compact and σ is finite dimensional, its associated cycle is simply given by $\mathcal{AC}\,(\sigma) = \dim\sigma \cdot [\{0\}]$. Recall $\mathcal{AC}\,(L(\sigma)) = \dim\sigma \cdot [\overline{\mathcal{O}_m}]$ from Theorem C. These formulas strongly indicate the following; there should be a correspondence between nilpotent orbits of the dual pairs, and it induces certain relation between associated cycles of representations in theta correspondence. An optimistic reflection suggests that, if $L(\sigma)$ is a theta lift of σ, then their associated varieties are related as

$$\mathcal{AC}\,(L(\sigma)) = \sum_i m_i[\overline{\mathcal{O}_i}] \quad \longleftrightarrow \quad \mathcal{AC}\,(\sigma) = \sum_i m_i[\overline{\mathcal{O}'_i}],$$

with the same multiplicity, where $\mathcal{O}_i \leftrightarrow \mathcal{O}'_i$ indicates the orbit correspondence. However we do not have an intuitive evidence of such a kind of correspondence other than the cases treated here.

Third, Theorem A (or K-type decompositions) suggests that we should "quantize" the orbit $\mathcal{O}_m$ to get an irreducible unitary representation $L(\mathbf{1}_{G_2})$, which certainly should be a unipotent representation. For this, it will be helpful to try the similar method exploited by Kostant-Brylinski in the case of the minimal orbit. However, this will require much more than what we have presented in this note.

Now let us explain each section briefly.

In §1, we define the associated cycles and other important invariants of representations in a general setting. After that, we collect their basic properties which will be needed later. In particular, in Lemma 1.1 and Theorem 1.4, we clarify the relationship between the associated cycles and the Bernstein degree (or the degree of the projectivised nilpotent cone); also, we recall the fact that the associated variety is the projection of the characteristic variety under the moment map (Lemma 1.6).

In §2, we briefly summarize known facts and examples of associated cycles of various types of representations. To see what is going on in this paper, §§1.3 and 2.4 will be extremely useful.

In §3, we review the properties of a reductive dual pair which we will need later. After an explicit description of the Fock realization of the Weil representation in §4, §5 is devoted to giving the complete description of the unitary lowest weight representations of G via theta correspondence.

In §6, we give a formula of K-type decomposition of the unitary lowest weight representations, using the branching coefficient of finite dimensional representations of compact groups. These formulas are well-known among experts, however, we need full detailed formulas in the following sections.

In §7, we study the geometry of nilpotent orbits in the case of Hermitian symmetric pair. Take a $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}^-$ and the unfolding $\psi : V \to \overline{\mathcal{O}_m}$ as above. We use ψ to study the ring of regular functions $\mathbb{C}[\overline{\mathcal{O}_m}]$ on $\overline{\mathcal{O}_m}$, and clarify its $K_{\mathbb{C}}$-module structure. This leads us an identification of $\deg \overline{\mathcal{O}_m}$ and the Bernstein degree of one of the smallest unitary lowest weight module attached to $\mathcal{O}_m$. As a result, Theorem A is proved. We also study the global sections of the half-form bundle over $\mathcal{O}_m$ in the tame cases.

In §8, we study a general theory of multiplicity free actions of a pair of reductive algebraic groups. We define the notions of degree and dimension of the space of covariants. The main result in this section is the formula of the degree and the dimension of covariants (Theorem 8.6).

In §9, we treat general unitary lowest weight representations of G, which are singular. By the results of §8, we prove Theorems B and C in this section.

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Notation: We denote the field of real (respectively, complex, quaternionic) numbers by $\mathbb{R}$ (respectively, $\mathbb{C}, \mathbb{H}$). If $\mathbb{K}$ is one of these fields, we use the following notation for subsets of matrices:

$M(n, m, \mathbb{K})$	the set of all $n \times m$ matrices,
$\mathrm{Sym}\,(n, \mathbb{K})$	the set of all symmetric matrices of size n,
$\mathrm{Alt}\,(n, \mathbb{K})$	the set of all alternating matrices of size n.

These subsets are abbreviated as $M_{n,m}, \mathrm{Sym}_n, \mathrm{Alt}_n$ respectively, if there is no confusion on the base fields. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, we also denote by skew-Her$\,(n, \mathbb{K})$ the set of all skew Hermitian matrices of size n. If τ_λ is an irreducible finite dimensional representation of $GL(m, \mathbb{C})$ (or $U(m)$) with highest weight λ, we often write it as $\tau_\lambda^{(m)}$, denoting the rank m of the group explicitly by the superscript.

1. Invariants of representations

1.1. A review on the commutative ring theory. — First of all, we review well-known results in the commutative ring theory, which we need in the subsequent sections. For more details of what are discussed here, we refer the readers to textbooks on the commutative ring theory, for example, [**10**], [**21**], [**33**].

Let V be an n-dimensional vector space over the field $\mathbb{C}$ and let $A := \mathbb{C}[V]$ be the ring of polynomials on V. For a finitely generated A-module M, the *support* $\operatorname{Supp} M$ of M is defined to be the set of prime ideals $\mathfrak{p}$ with $M_{\mathfrak{p}} \neq 0$. Since M is finitely generated, $\operatorname{Supp} M$ coincides with the Zariski closure of $\operatorname{Ann} M := \{a \in A \mid aM = 0\}$, which is denoted by $\mathcal{V}(M)$. We often identify $\mathcal{V}(M)$ with the affine variety

$$\mathcal{V}(M) \cap \text{m-Spec}\, A = \{x \in V \mid p(x) = 0 \ \ (\forall p \in \operatorname{Ann} M)\}.$$

Let A_n be the set of homogeneous polynomials of degree n. By the natural grading $A = \oplus_{n=0}^{\infty} A_n$, A is a graded $\mathbb{C}$-algebra. Let $M = \oplus_{n=0}^{\infty} M_n$ be a finitely generated graded A-module. As usual, we denote the Poincaré series by $P(M;t)$. It is well-known that there exists a unique polynomial $Q(t)$ and a non-negative integer d such that

$$P(M;t) = \sum_{n=0}^{\infty} (\dim M_n) t^n = \frac{Q(t)}{(1-t)^d}, \quad Q(1) \neq 0. \tag{1.1}$$

It turns out that $Q(1)$ is a positive integer. By the expression (1.1), we know that $\dim M_n$ is a polynomial in n for sufficiently large n, and it is written as

$$\dim M_n = \frac{Q(1)}{(d-1)!} n^{d-1} + (\text{lower order terms of } n).$$

Note that the integer d is the dimension of $\mathcal{V}(M)$. The integer $Q(1)$ is called the *multiplicity of* M, and we denote it by $\mathbf{m}(M)$.

A prime ideal $\mathfrak{P} \in \operatorname{Spec} A$ is called an associated prime of M if $\mathfrak{P}$ is an annihilator of some non-zero element of M. The set of associated primes is denoted by $\operatorname{Ass} M$. It is easy to see that $\operatorname{Ass} M \subset \operatorname{Supp} M$. The set of minimal elements of $\operatorname{Ass} M$ and that of $\operatorname{Supp} M$ coincide, and they form a finite set. Let $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$ be the set of minimal primes in $\operatorname{Supp} M$, and let $\mathcal{V}(M) = \cup_{i=1}^{r} C_i$ be the corresponding irreducible decomposition of the variety $\mathcal{V}(M)$.

Choose $\mathfrak{Q}^1 \in \operatorname{Ass} M$. Then there exists a submodule $M^1 \subset M$ such that $M^1 \simeq A/\mathfrak{Q}^1$. By induction, there exists a finite sequence $0 = M^0 \subset M^1 \subset \cdots \subset M^l = M$ such that $M^k/M^{k-1} \simeq A/\mathfrak{Q}^k$ for some $\mathfrak{Q}^k \in \operatorname{Spec} A$ $(k = 1, 2, \ldots, l)$. It is not hard to check that the integer

$$\operatorname{mult}_{\mathfrak{P}}(M) := \#\{\mathfrak{Q}^k \mid \mathfrak{Q}^k = \mathfrak{P}\}, \quad \mathfrak{P} : \text{minimal prime}$$

is independent of the choice of the sequence $\{M^k\}_k$. This integer is called the *multiplicity of* M *at* $\mathfrak{P}$. Note that $\operatorname{mult}_{\mathfrak{P}}(M)$ is reinterpreted as the length of Artinian $A_{\mathfrak{P}}$-module $M_{\mathfrak{P}}$. By the correspondence of minimal $\mathfrak{P} \in \operatorname{Ass} M$ and the irreducible component C of $\mathcal{V}(M)$, we also denote the multiplicity by $\operatorname{mult}_C(M)$.

As a refinement of $\operatorname{Supp} M$, we consider the formal linear combination of the minimal primes $\mathfrak{P}_i$ (or irreducible components C_i) with coefficients $\operatorname{mult}_{\mathfrak{P}_i}(M) =$

$\mathrm{mult}_{C_i}(M)$,

$$\underline{\mathrm{Supp}}\, M := \sum_i \mathrm{mult}_{\mathfrak{P}_i}(M)\, [\mathfrak{P}_i] = \sum_i \mathrm{mult}_{C_i}(M)\, [C_i].$$

More generally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on an algebraic variety X. We can refine $\mathrm{Supp}\,\mathcal{F}$ analogously. For any irreducible component C of the support of $\mathcal{F}$, the rank of the module $\mathcal{F}$ at a generic point of C is a well-defined positive integer $\mathrm{mult}_C(\mathcal{F})$. This is called the multiplicity of C in the support of $\mathcal{F}$. Then we consider the formal linear combination of the components C of $\mathrm{Supp}(\mathcal{F})$ with coefficients $\mathrm{mult}_C(\mathcal{F})$,

$$\underline{\mathrm{Supp}}(\mathcal{F}) = \sum_C \mathrm{mult}_C(\mathcal{F})\, [C].$$

The multiplicity of M can be obtained from $\underline{\mathrm{Supp}}\, M$. Let $\deg C$ be the degree of the variety C, i.e. $\deg C = \mathbf{m}(A/\mathfrak{P})$ (see, e.g., [**16**]). Since the Poincaré series is additive, $\mathbf{m}(M)$ is the sum of $\mathbf{m}(M^k/M^{k-1})$'s with $\dim \mathcal{V}(M^k/M^{k-1}) = \dim \mathcal{V}(M)$. By the definition of the sequence $\{M^k\}_k$ and the multiplicity $\mathrm{mult}_{\mathfrak{P}}(M)$, we have

Lemma 1.1. —

$$\mathbf{m}(M) = \sum_{\substack{i \\ \dim C_i = \dim \mathcal{V}(M)}} \mathrm{mult}_{C_i}(M) \deg C_i.$$

Remark 1.2. — The notion of degree is usually defined for projective varieties. In our case, we can projectivise $\mathcal{V}(M)$ and its irreducible components since $\mathrm{Ann}\, M$ is graded. Then $\deg C_i$ should be interpreted as the degree of the projectivised variety.

1.2. Invariants of $U(\mathfrak{g})$-modules. — In this subsection, we introduce invariants of representations of Lie algebras after [**49**], [**51**]. These invariants are main objects of this paper.

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. We denote by $U_n(\mathfrak{g})$ the finite dimensional subspace of $U(\mathfrak{g})$, spanned by products of at most n-elements of $\mathfrak{g}$. Then $\{U_n(\mathfrak{g})\}_{n=0}^{\infty}$ is a filtration of $U(\mathfrak{g})$, called the *standard filtration*. By the Poincaré-Birkhoff-Witt (PBW) theorem, the associated graded algebra $\mathrm{gr}\, U(\mathfrak{g}) = \oplus_{n=0}^{\infty} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$.

Let V be a $U(\mathfrak{g})$-module. A chain $0 = V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V$, where the V_n's are subspaces of V, is called a *filtration of* V if it satisfies the following conditions:

$$\bigcup_{n=0}^{\infty} V_n = V, \qquad U_n(\mathfrak{g}) V_m \subset V_{n+m}, \qquad \dim V_n < \infty.$$

By the second condition, the graded object

$$\mathrm{gr}\, V = \bigoplus_{n=0}^{\infty} \mathrm{gr}_n V, \quad \mathrm{gr}_n V := V_n/V_{n-1}$$

has the structure of a graded $S(\mathfrak{g})$-module. A filtration is called *good* if it also satisfies

$$U_n(\mathfrak{g})V_m = V_{n+m} \quad \text{(for all } m \text{ sufficiently large, all } n \geq 0). \tag{1.2}$$

In this case, V is a finitely generated $U(\mathfrak{g})$-module and $\operatorname{gr} V$ is a finitely generated $S(\mathfrak{g})$-module. Conversely, if V is finitely generated, we can construct a good filtration by choosing a finite dimensional generating subspace V_0 and by putting $V_n = U_n(\mathfrak{g})V_0$.

Regarding the symmetric algebra $A = S(\mathfrak{g})$ as the polynomial ring on the dual space $\mathfrak{g}^*$, we define several invariants of V using those defined via commutative ring theory.

Definition 1.3. — For a finitely generated $U(\mathfrak{g})$-module V, we define the *associated variety* $\mathcal{AV}(V)$, the *associated cycle* $\mathcal{AC}(V)$, the *Gelfand-Kirillov dimension* $\operatorname{Dim} V$, and the *Bernstein degree* $\operatorname{Deg} V$ by

$$\begin{aligned} \mathcal{AV}(V) &= \mathcal{V}(\operatorname{gr} V), & \mathcal{AC}(V) &= \underline{\operatorname{Supp}}(\operatorname{gr} V), \\ \operatorname{Dim} V &= \dim \mathcal{AV}(V), & \operatorname{Deg} V &= \mathbf{m}(\operatorname{gr} V) \end{aligned}$$

respectively. They are independent of the choice of good filtrations of V, and therefore well-defined for V.

For an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

of finitely generated $U(\mathfrak{g})$-modules, we have $\operatorname{Dim} V_2 = \max\{\operatorname{Dim} V_1, \operatorname{Dim} V_3\}$, and

$$\mathcal{AV}(V_2) = \mathcal{AV}(V_1) \cup \mathcal{AV}(V_3). \tag{1.3}$$

Note that the associated cycle is not additive in general, i.e., $\mathcal{AC}(V_2) \neq \mathcal{AC}(V_1) + \mathcal{AC}(V_3)$. If we write

$$c_d(V) = \begin{cases} \operatorname{Deg} V & \text{if } d = \operatorname{Dim} V, \\ 0 & \text{if } d > \operatorname{Dim} V, \end{cases}$$

then the Bernstein degree becomes additive in the sense that

$$\operatorname{Deg} V_2 = c_{\operatorname{Dim} V_2}(V_2) = c_{\operatorname{Dim} V_2}(V_1) + c_{\operatorname{Dim} V_2}(V_3).$$

The right hand side is equal to $\operatorname{Deg} V_1 + \operatorname{Deg} V_3$ if $\operatorname{Dim} V_1 = \operatorname{Dim} V_3$.

1.3. The structure of invariants of Harish-Chandra modules. — The associated variety of a module (with some assumption, of course) over a reductive Lie algebra $\mathfrak{g}$ is contained in the nilpotent cone in $\mathfrak{g}^*$. Moreover, if it is a Harish-Chandra $(\mathfrak{g}, K)$-module, the associated variety has a $K_{\mathbb{C}}$-orbit structure. In this subsection, we shall review these well-known results.

Let G be a connected reductive group over $\mathbb{R}$ and $\mathfrak{g}_0$ its Lie algebra. Take a maximal compact subgroup $K \subset G$ and let $K_{\mathbb{C}}$ be its complexification. Denote by $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition associated to K and by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its complexification. For a Harish-Chandra $(\mathfrak{g}, K)$-module $\mathcal{H}$, we choose a finite dimensional K-invariant generating subspace $\mathcal{H}_0$ and define a filtration by $\mathcal{H}_n = U_n(\mathfrak{g})\mathcal{H}_0$. Then the graded object $\operatorname{gr} \mathcal{H}$ has compatible $S(\mathfrak{g})$- and $K_{\mathbb{C}}$-actions.

By the compatibility of $\mathfrak{g}$- and $K_{\mathbb{C}}$-actions, $\mathcal{AV}(\mathcal{H})$ is invariant under the action of $K_{\mathbb{C}}$ and $\mathfrak{k}$ acts on $\operatorname{gr}\mathcal{H}$ trivially. It follows that $\mathcal{AV}(\mathcal{H})$ is a $K_{\mathbb{C}}$-invariant subvariety in $(\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{p}$.

Fix a connected algebraic group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. The algebra $U(\mathfrak{g})^{G_{\mathbb{C}}}$ of $\operatorname{Ad}(G_{\mathbb{C}})$-invariants in $U(\mathfrak{g})$ is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, since $G_{\mathbb{C}}$ is connected. Filter the algebra $U(\mathfrak{g})^{G_{\mathbb{C}}}$ by the standard filtration of $U(\mathfrak{g})$, then $\operatorname{gr} U(\mathfrak{g})^{G_{\mathbb{C}}}$ is isomorphic to $S(\mathfrak{g})^{G_{\mathbb{C}}}$, the algebra of $\operatorname{Ad}(G_{\mathbb{C}})$-invariants in $S(\mathfrak{g})$. Since any irreducible $U(\mathfrak{g})$-module is annihilated by a maximal ideal in $U(\mathfrak{g})^{G_{\mathbb{C}}}$, any $U(\mathfrak{g})$-module of finite length is annihilated by the product of a finite number of maximal ideals in $U(\mathfrak{g})^{G_{\mathbb{C}}}$. Such a product is of finite codimension in $U(\mathfrak{g})^{G_{\mathbb{C}}}$. Therefore, the radical of the graded object of this product is the ideal $S^+(\mathfrak{g})^{G_{\mathbb{C}}}$, the set of invariant polynomials without constant term. This argument implies that the associated variety $\mathcal{AV}(\mathcal{H})$ is contained in the zero set $\mathcal{V}(S^+(\mathfrak{g})^{G_{\mathbb{C}}})$ of $S^+(\mathfrak{g})^{G_{\mathbb{C}}}$. Note that $\mathcal{V}(S^+(\mathfrak{g})^{G_{\mathbb{C}}})$ coincides with the set $\mathcal{N}^*$ of nilpotent elements in $\mathfrak{g}^*$, since $G_{\mathbb{C}}$ is connected.

Consequently, $\mathcal{AV}(\mathcal{H})$ is a union of $K_{\mathbb{C}}$-orbits in $\mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^* \simeq \mathcal{N}_{\mathfrak{p}}$, the set of nilpotent elements in $\mathfrak{p}$. By a theorem of Kostant-Rallis, $\mathcal{N}_{\mathfrak{p}}$ is a finite union of $K_{\mathbb{C}}$-orbits. Summarizing the above discussion and the results of many contributors, we have the following well-known theorem.

Theorem 1.4. — *If $\mathcal{H}$ is a Harish-Chandra $(\mathfrak{g}, K)$-module, then the associated variety $\mathcal{AV}(\mathcal{H})$ is a finite union of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$. Moreover, if $\mathcal{H}$ is irreducible, we have the following.*

(1) *There exist nilpotent $K_{\mathbb{C}}$-orbits $\{C_i\} \subset \mathcal{N}_{\mathfrak{p}}$ with dimension equal to* $\operatorname{Dim}\mathcal{H}$ *such that*

$$\mathcal{AV}(\mathcal{H}) = \bigcup_{i=1}^{l} \overline{C_i}. \tag{1.4}$$

(2) *Denote the associated cycle as $\mathcal{AC}(\mathcal{H}) = \sum_i m_i[\overline{C_i}]$. Then the Bernstein degree is given by*

$$\operatorname{Deg}\mathcal{H} = \sum_{i=1}^{l} m_i \deg \overline{C_i}. \tag{1.5}$$

(3) *Let $I = I_{\mathcal{H}} \subset U(\mathfrak{g})$ be the associated primitive ideal. Then $\mathcal{AV}(U(\mathfrak{g})/I)$ is the closure of a single nilpotent $G_{\mathbb{C}}$-orbit $C_{\mathcal{H}}$, and for any i, the $G_{\mathbb{C}}$-orbit through C_i coincides with $C_{\mathcal{H}}$. In fact, $C_{\mathcal{H}} \cap \mathfrak{p}$ decomposes into a finite union of equidimensional nilpotent $K_{\mathbb{C}}$-orbits, and $\{C_i\}$ is a subset of its irreducible components:*

$$C_{\mathcal{H}} \cap \mathfrak{p} \supset C_1, \ldots, C_l. \tag{1.6}$$

Remark 1.5. — Take $\lambda \in C_i$ and denote by $K_{\mathbb{C}}(\lambda)$ the fixed subgroup of $K_{\mathbb{C}}$ at λ. The multiplicity m_i in (2) can be interpreted as the dimension of a certain representation of $K_{\mathbb{C}}(\lambda)$. For this, we refer to [**51**, Definition 2.12].

1.4. Invariants of $\mathcal{D}_X$-modules. — The relation between the associated varieties (associated cycles) and the characteristic varieties (characteristic cycles) is discussed in [**3**]. First, we recall the definition of the characteristic varieties and the characteristic cycles, which is analogous to that of the associated variety and its cycle for $\mathfrak{g}$-module given in § 1.2.

Let X be a smooth algebraic variety over an algebraically closed field $\mathbb{C}$. We denote by $\mathcal{D}_X$ the sheaf of (algebraic) differential operators on X. On $\mathcal{D}_X$, we have a natural increasing filtration by the $\mathcal{O}_X$-submodules $\mathcal{D}_X(n)$, the subsheaf of all differential operators of order $\leq n$;

$$0 = \mathcal{D}_X(-1) \subset \mathcal{D}_X(0) = \mathcal{O}_X \subset \mathcal{D}_X(1) \subset \cdots \subset \mathcal{D}_X.$$

The associated graded sheaf

$$\operatorname{gr} \mathcal{D}_X = \bigoplus_{n=0}^{\infty} \operatorname{gr}_n \mathcal{D}_X, \qquad \operatorname{gr}_n \mathcal{D}_X = \mathcal{D}_X(n)/\mathcal{D}_X(n-1)$$

is naturally identified with the direct image sheaf $\pi_*(\mathcal{O}_{T^*X})$, where $\pi : T^*X \to X$ is the cotangent bundle of X, and $\mathcal{O}_{T^*X}$ is the structure sheaf of T^*X.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then there is a good filtration

$$0 = \mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}.$$

The corresponding graded module is defined by

$$\operatorname{gr} \mathcal{M} = \bigoplus_{n=0} \operatorname{gr}_n \mathcal{M}, \qquad \operatorname{gr}_n \mathcal{M} = \mathcal{M}_n/\mathcal{M}_{n-1}.$$

Then $\operatorname{gr} \mathcal{M}$ is coherent over $\pi_*(\mathcal{O}_{T^*X})$. The support of $\operatorname{gr} \mathcal{M}$ as a module on T^*X, more precisely, the support of $\mathcal{O}_{T^*X} \otimes_{\operatorname{gr} \mathcal{D}_X} \operatorname{gr} \mathcal{M}$, is called the *characteristic variety* of $\mathcal{M}$. This is a closed conic algebraic subvariety of the cotangent bundle T^*X, which is usually denoted by

$$\operatorname{Ch}(\mathcal{M}) = \operatorname{Supp}(\operatorname{gr} \mathcal{M}).$$

The variety does not depend on the choice of a good filtration. As a refinement of $\operatorname{Ch}(\mathcal{M})$, we define the *characteristic cycle* of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ by

$$\underline{\operatorname{Ch}}(\mathcal{M}) = \underline{\operatorname{Supp}}(\operatorname{gr} \mathcal{M}).$$

The characteristic cycle is also independent of the choice of a good filtration.

From now on in this subsection, let G be a reductive algebraic group over $\mathbb{C}$ and let X be the set of Borel subgroups of G. Then it is known that X is a complete G-homogeneous variety, and the Lie algebra $\mathfrak{g}$ of G acts on X by vector fields on X. This gives a Lie algebra homomorphism

$$\mathfrak{g} \to D(X),$$

where $D(X) = \Gamma(X, \mathcal{D}_X)$ denotes the set of all global sections of the sheaf $\mathcal{D}_X$ on X. This map extends to an algebra homomorphism

$$\psi : U(\mathfrak{g}) \to D(X),$$

which is known to be surjective. With the natural filtrations, $\operatorname{gr} U(\mathfrak{g})$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$, while $\operatorname{gr} D(X)$ to the set of global sections

$\mathbb{C}[T^*X]$ of (algebraic) holomorphic functions on the cotangent bundle T^*X. Since the map ψ is compatible with the natural filtrations, we have the associated graded ring homomorphism

$$\phi = \operatorname{gr}\psi : S(\mathfrak{g}) \to \mathbb{C}[T^*X].$$

The map ϕ gives rise to the moment map

$$\mu : T^*X \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$. It is known that the image of μ is normal and that the map μ is birational onto its image. The moment map is the key to give a relation between the characteristic variety and the associated variety.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then the set of all global sections $M = \Gamma(X, \mathcal{M})$ is a module over $D(X) = \Gamma(X, \mathcal{D}_X)$. Using the algebra homomorphism ψ, a $D(X)$-module is considered as a $\mathfrak{g}$-module. Then M is a finitely generated $\mathfrak{g}$-module with the trivial central character. Conversely, any finitely generated $\mathfrak{g}$-module M with the trivial central character can be obtained in this manner from a coherent $\mathcal{D}_X$-module $\mathcal{M}$. Indeed, $\mathcal{M}$ is obtained by, so called, the localization such as $\mathcal{M} = \mathcal{D}_X \otimes_{U(\mathfrak{g})} M$ using the homomorphism ψ.

Lemma 1.6. — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Consider $M = \Gamma(X, \mathcal{M})$ as a $\mathfrak{g}$-module.*

(1) *The associated variety of M is the image of the characteristic variety of $\mathcal{M}$ under the moment map:*

$$\mathcal{AV}(M) = \mu(\mathrm{Ch}(\mathcal{M})).$$

(2) *Suppose, moreover, that $\mathcal{M}$ has a good filtration $\{\mathcal{M}_j\}_j$ such that $H^1(X, \mathcal{M}_j) = 0$. We denote the direct image under the moment map of the $\mathcal{O}_{T^*X}$-module $\operatorname{gr}\mathcal{M}$ by $\mu_*(\operatorname{gr}\mathcal{M})$, which is a coherent $\mathcal{O}_{\mathfrak{g}^*}$-module. Then the associated cycle is described by the cycle of this module*

$$\mathcal{AC}(M) = \underline{\mathrm{Supp}}(\mu_*(\operatorname{gr}\mathcal{M})),$$

where the definition of the cycle of $\mathcal{O}_{\mathfrak{g}^*}$-module is given in §1.1.

More general statement would be found in Theorem 1.9 and Remark to Lemma 1.6 in [**3**]. The condition of the vanishing of the first cohomology appearing in (2) of the lemma holds for sufficiently regular infinitesimal characters, due to a result of Serre. See Appendix A of [**3**], for details.

2. Known results and examples

In this section, we summarize known results and examples of the invariants defined in §1. Some of them are immediately obtained from the definition, others are non-trivial.

2.1. Finite dimensional representation. — For a finite dimensional representation V of a complex Lie algebra $\mathfrak{g}$, we may take $V_0 = V$ and consequently $V_n = V$ for all $n \geq 0$. Then the Poincaré series is a constant $\dim V$, and we have

$$\operatorname{Dim} V = 0 \quad \text{and} \quad \operatorname{Deg} V = \dim V.$$

From this, we conclude that the associated variety of V is $\{0\}$, and the associated cycle equals $\mathcal{AC}\,(V) = (\dim V) \cdot [\{0\}]$.

2.2. Generalized Verma module. — Let $\mathfrak{q} = \mathfrak{l} + \bar{\mathfrak{u}}$ be the Levi decomposition of a parabolic subalgebra $\mathfrak{q}$ of a complex reductive Lie algebra $\mathfrak{g}$, where $\mathfrak{l}$ is a Levi subalgebra and $\bar{\mathfrak{u}}$ is the nilpotent radical of $\mathfrak{q}$. Denote by $\mathfrak{u}$ the opposite nilpotent Lie algebra to $\bar{\mathfrak{u}}$. Take an irreducible finite dimensional representation τ_λ of $\mathfrak{l}$ with the highest weight λ and extend it to a representation of $\mathfrak{q}$ trivially. The generalized Verma module $M(\lambda)$ is defined by $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} \tau_\lambda$.

Proposition 2.1. — *The invariants for the generalized Verma module $M(\lambda)$ are*

$$\operatorname{Dim} M(\lambda) = \dim \mathfrak{u} = \dim \bar{\mathfrak{u}}, \qquad \operatorname{Deg} M(\lambda) = \dim \tau_\lambda, \tag{2.7}$$

$$\mathcal{AV}\,(M(\lambda)) = \bar{\mathfrak{u}} \quad \textit{and} \qquad \mathcal{AC}\,(M(\lambda)) = (\dim \tau_\lambda)[\bar{\mathfrak{u}}]. \tag{2.8}$$

Here, we identified $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form.

Proof. — By the PBW theorem, $M(\lambda) = U(\mathfrak{u}) \otimes_{\mathbb{C}} \tau_\lambda$ as a vector space and $M(\lambda)_n := U_n(\mathfrak{u}) \otimes_{\mathbb{C}} \tau_\lambda$ $(n = 0, 1, 2, \dots)$ defines a good filtration of $M(\lambda)$. We denote the associated graded module by $\operatorname{gr} M(\lambda)$. Since

$$\dim \operatorname{gr}_n M(\lambda) = (\dim \tau_\lambda) \times \binom{n + \dim \mathfrak{u} - 1}{\dim \mathfrak{u} - 1},$$

we immediately conclude that $\operatorname{Dim} M(\lambda) = \dim \mathfrak{u}$ and $\operatorname{Deg} M(\lambda) = \dim \tau_\lambda$.

Next, we shall calculate $\mathcal{AC}\,(M(\lambda))$. Since $\mathfrak{q}$ is contained in $\operatorname{Ann}_{S(\mathfrak{g})} \operatorname{gr} M(\lambda)$ and the intersection $S(\mathfrak{u}) \cap \operatorname{Ann}_{S(\mathfrak{g})} \operatorname{gr} M(\lambda)$ is $\{0\}$, $\operatorname{Ann}_{S(\mathfrak{g})} \operatorname{gr} M(\lambda)$ coincides with $S(\mathfrak{g})\mathfrak{q}$. Then

$$\mathcal{AV}\,(M(\lambda)) = \{x \in \mathfrak{g}^* \mid \langle x, \mathfrak{q} \rangle = \{0\}\} \simeq \bar{\mathfrak{u}}.$$

Moreover, since $\bar{\mathfrak{u}} \simeq \mathbb{C}^{\dim \bar{\mathfrak{u}}}$ is irreducible and its degree is one, the multiplicity is $\dim \tau_\lambda$ by Lemma 1.1. $\square$

2.3. Lowest weight module. — We use the same notation as in the previous subsection.

Let V be a $\mathfrak{q}$-lowest weight $U(\mathfrak{g})$-module, i.e. there exists an irreducible finite dimensional $\mathfrak{l}$-submodule V_0 in V such that $\bar{\mathfrak{u}}$ acts trivially on it and V is generated by it. Let λ be the highest weight of V_0. By the universality of the generalized Verma module, there exists a unique surjective $U(\mathfrak{g})$-homomorphism

$$\Phi : M(\lambda) \twoheadrightarrow V.$$

By this homomorphism, a good filtration on V is induced from that of $M(\lambda)$. By (1.3) and (2.8), we have

$$\mathcal{AV}(V) \subset \bar{\mathfrak{u}}. \tag{2.9}$$

2.4. Hermitian symmetric case. — Let (G, K) be an irreducible Hermitian symmetric pair. We use the notation in § 1.3. The adjoint representation of K on $\mathfrak{p}$ decomposes into two irreducible components $\mathfrak{p}^{\pm}$. Since $\mathfrak{q} := \mathfrak{k} + \mathfrak{p}^-$ is a maximal parabolic subalgebra of $\mathfrak{g}$, we can apply the results in §§ 1.3 and 2.2 for a $\mathfrak{q}$-lowest weight module. By (2.9) and Theorem 1.4, the associated variety of a $\mathfrak{q}$-lowest weight $(\mathfrak{g}, K)$-module is a finite union of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$.

In particular, since the $(\mathfrak{g}, K)$-module of the holomorphic discrete series is a generalized Verma module, the invariants for it are given by (2.7) and (2.8), where τ_λ is the minimal K-type and $\bar{\mathfrak{u}} = \mathfrak{p}^-$. Namely we have

Proposition 2.2. — *Let π_λ be a holomorphic discrete series representation of G with the minimal K-type τ_λ. Then invariants of π_λ are given as*

$$\operatorname{Dim} \pi_\lambda = \dim \mathfrak{p}^- = \frac{1}{2} \dim G/K, \qquad \operatorname{Deg} \pi_\lambda = \dim \tau_\lambda, \tag{2.10}$$

$$\mathcal{AV}(\pi_\lambda) = \mathfrak{p}^- \quad \textit{and} \qquad \mathcal{AC}(\pi_\lambda) = (\dim \tau_\lambda)[\mathfrak{p}^-]. \tag{2.11}$$

Let us consider the Poincaré series of a $\mathfrak{q}$-lowest weight module V. Let Z be the center of K and let $\mathfrak{z}_0$ be its Lie algebra. Under our setting, every element of Z acts on $\mathfrak{p}^{\pm}$ by a non-trivial scalar and it acts on the minimal K-type of V also by a scalar.

Choose a base H of $\mathfrak{z}_0$ and denote by α the scalar $\operatorname{ad}(H)|_{\mathfrak{p}^+}$. Let $h(s) := \exp sH \in K$. The action of $h(s)$ on V gives the Poincaré series of V. More precisely,

Proposition 2.3. — *The Poincaré series of a $\mathfrak{q}$-lowest weight module V is*

$$P(\operatorname{gr} V; t) = t^{-n_0} \left(\operatorname{trace} h(s)|_V\right),$$

where $t = e^{\alpha s}$ and n_0 is the scalar by which H acts on the minimal K-type of V.

Proof. — First, we consider the generalized Verma module $M(\lambda)$. The action of $h(s)$ on $M(\lambda)_n$ is a scalar $e^{(n+n_0)\alpha s}$. By the definition of the Poincaré series, we have $P(\operatorname{gr} M(\lambda); t) = t^{-n_0} \left(\operatorname{trace} h(s)|_{M(\lambda)}\right)$. Using the universality of the Verma module, we obtain the Poincaré series of a lowest weight module V in the same way. □

2.5. Discrete series of real rank one groups. — For the discrete series representations of real rank one groups, the associated cycles are explicitly obtained by Chang [**8**].

By many contributors, the associated variety of a discrete series is well-known. Especially, it is a closure of a single $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$, and irreducible (see Theorem 1.4). Then the problem reduces to the determination of the multiplicity. Using the relation between the associated cycle and the characteristic cycle (Lemma 1.6), he calculated it by investigating the fiber of the moment map.

For the explicit value of the multiplicity, we refer to his paper.

2.6. Large representation. — Let G be a real reductive Lie group and let $G = KA_mN_m$, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_{m,0} + \mathfrak{n}_{m,0}$ be the Iwasawa decomposition of G and $\mathfrak{g}_0 := \operatorname{Lie} G$, respectively.

For a Harish-Chandra $(\mathfrak{g}, K)$-module V, it is known that the Gelfand-Kirillov dimension is at most $\dim \mathfrak{n}_{m,0}$ ([**49**]). We call V *large* if $\operatorname{Dim} V = \dim \mathfrak{n}_{m,0}$. In this case, V has Whittaker models and the dimension of models coincides with the Bernstein degree of V.

To state more precisely, we need some notation. Let $\psi : N_m \to \mathbb{C}^\times$ be a unitary character. We denote the differential character of $\mathfrak{n}_{m,0}$ by the same symbol ψ. Then ψ is identified with an element of $\sqrt{-1}(\mathfrak{n}_{m,0}/[\mathfrak{n}_{m,0}, \mathfrak{n}_{m,0}])^*$. We call ψ *admissible* if the coadjoint M_mA_m-orbit of ψ is open in $(\mathfrak{n}_{m,0}/[\mathfrak{n}_{m,0}, \mathfrak{n}_{m,0}])^*$. Here, M_m is the centralizer of A_m in K. For an admissible ψ, we define the space of dual Whittaker vectors $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ by

$$\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V) := \{v^* \in V^* \mid Xv^* = \psi(X)v^* \ \ (\forall X \in \mathfrak{n}_{m,0})\},$$

where V^* is the dual space of V.

Theorem 2.4 **([36])**. — *The space* $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ *is not zero if and only if* $\operatorname{Dim} V = \dim \mathfrak{n}_{m,0}$. *In this case, the dimension of* $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ *equals* $\operatorname{Deg} V$.

If V is a principal series representation, the dimension of $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ is obtained by Kostant (quasi-split case, [**30**]) and Lynch (non-quasi-split case, [**32**]). Thus by the above theorem, we know the Bernstein degree of V:

Theorem 2.5 **([30], [32])**. — *The principal series representation* $\operatorname{Ind}^G_{M_mA_mN_m}(\sigma \otimes e^\nu \otimes 1)$ *is large, and the Bernstein degree is* $\#W(\mathfrak{g}_0, \mathfrak{a}_{m,0}) \cdot \dim \sigma$, *where* $W(\mathfrak{g}_0, \mathfrak{a}_{m,0})$ *is the little Weyl group.*

Remark 2.6. — The associated variety of a principal series representation is a finite union of the closure of regular nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$. Let $\{\mathcal{N}_1, \ldots, \mathcal{N}_l\} \subset \mathfrak{p}$ be the set of all regular nilpotent $K_{\mathbb{C}}$-orbits. Then we have

$$\sum_{i=1}^{l} \deg \overline{\mathcal{N}_i} = \#W(\mathfrak{g}_0, \mathfrak{a}_{m,0})$$

(see [**31**]). Since $\deg \overline{\mathcal{N}_i} = \deg \overline{\mathcal{N}_j}$, we see that $\deg \overline{\mathcal{N}_i} = \#W(\mathfrak{g}_0, \mathfrak{a}_{m,0})/l$.

There are explicit calculations of Whittaker models of some low rank groups. For the following representations, the Whittaker models are explicitly determined.

(1) Large discrete series representations of $Sp(2,\mathbb{R})$ (by Oda [**38**]).

(2) Large discrete series representations of $SU(2,2)$ (by Yamashita [**53**] and Hayata-Oda [**19**]).

(3) The generalized principal series representation $\mathrm{Ind}_{P_J}^{G}(\sigma \otimes e^{\nu+\rho_J} \otimes 1)$ of $G = Sp(2,\mathbb{R})$ (by Hayata [**18**]). Here, $P_J = M_J A_J N_J$ is the Jacobi parabolic subgroup of G and σ is a discrete series representation of $M_J \simeq \mathbb{C}^\times \times SU(1,1)$.

(4) Large discrete series representations of $SU(n,1)$ and $Spin(2n,1)$ (by Taniguchi [**46**]).

From these calculations, we know their Bernstein degrees. The Bernstein degrees of (1)–(3) are all four. Those of (4) are twice the multiplicities, which are obtained by Chang (see § 2.5). In other words, the degrees of the associated variety of large discrete series representations of $SU(n,1)$ and $Spin(2n,1)$ are two (cf. Lemma 1.1).

2.7. Minimal representation. — In this subsection, we will give Bernstein degrees of so-called minimal representations. Here we only consider *non-Hermitian* symmetric space G/K, though the arguments below equally works well for general situations.

If G/K is non-Hermitian, G has a minimal representation if and only if G/K is in the following list.

– Classical case : $SO(p,q)/SO(p) \times SO(q)$ where $p \geq q \geq 3, p+q \in 2\mathbb{Z}$ or $p \in 2\mathbb{Z}, q = 3$.

– Exceptional case : The following 8 cases.

$F_{4,4}/Sp(3) \times SU(2)$	$G_2/SO(4)$	$E_{6,4}/SU(2) \times SU(6)$	$E_{6,6}/Sp(4)$
$E_{7,4}/Spin(12) \times SU(2)$	$E_{7,7}/SU(8)$	$E_{8,4}/E_7 \times SU(2)$	$E_{8,8}/Spin(16)$

Take the minimal nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\min} \subset \mathfrak{g}$. Then in this case $\mathcal{O}_{\min} \cap \mathfrak{p} =: Y$ is a single nilpotent $K_{\mathbb{C}}$-orbit, which is minimal among non-zero nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$ with respect to the closure relation.

***Theorem 2.7* (Vogan).** — *Let $\pi_{\min}$ be a minimal representation of G. Then there exists some weight ν such that*

$$\pi_{\min}|_K \simeq \bigoplus_{m \geq 0} \tau_{m\psi+\nu},$$

where ψ is the highest weight of $\mathfrak{p}$ (= the highest root), and $\tau_{m\psi+\nu}$ is the irreducible representation of K with highest weight $m\psi + \nu$.

***Remark 2.8*.** — The weight ν is the highest weight of the minimal K-type of $\pi_{\min}$. For an explicit description of ν, we refer to Table 1 of [**5**] and the references cited there.

Put $\Delta_c^+(\psi) = \{\alpha \in \Delta_c^+ \mid \langle \psi, \alpha \rangle \neq 0\}$, where Δ_c^+ denotes the totality of positive compact roots. For $\alpha \in \Delta_c^+$, note that $\langle \psi, \alpha \rangle \neq 0$ if and only if $2\langle \psi, \alpha \rangle / \langle \psi, \psi \rangle = 1$ ([**27**, Lemma 2.2]).

Proposition 2.9. — *With the above notation, we have*

$$\begin{aligned}
&\operatorname{Dim}(\pi_{\min}) = \#\Delta_c^+(\psi) + 1 = \dim_{\mathbb{C}} \mathcal{O}_{\min}/2 = \dim_{\mathbb{C}} Y, \\
&\operatorname{Deg}(\pi_{\min}) = \deg \overline{Y} = \big(\#\Delta_c^+(\psi)\big)! \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle\psi,\psi\rangle}{2\langle\rho_c,\alpha\rangle}, \\
&\mathcal{AC}(\pi_{\min}) = [\overline{Y}].
\end{aligned}$$

Proof. — From the explicit description of ν (cf. [**5**, Table 1]), we conclude that $\langle\nu,\alpha\rangle = 0$ holds for each positive compact root $\alpha \notin \Delta_c^+(\psi)$. Also we know a good filtration of $(\pi_{\min}, V)$ is given by

$$V_n = \bigoplus_{m\le n} \tau_{m\psi+\nu}$$

(see [**50**]). Put $d = \#\Delta_c^+(\psi) + 1$. By Weyl's dimension formula, we calculate the dimension of $\operatorname{gr}_n V$ as

$$\begin{aligned}
\dim \operatorname{gr}_n V = \dim \tau_{n\psi+\nu} &= \prod_{\alpha\in\Delta_c^+} \frac{\langle n\psi+\nu+\rho_c,\alpha\rangle}{\langle\rho_c,\alpha\rangle} \\
&= n^{d-1} \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle\psi,\alpha\rangle}{\langle\rho_c,\alpha\rangle} \prod_{\alpha\notin\Delta_c^+(\psi)} \frac{\langle\nu+\rho_c,\alpha\rangle}{\langle\rho_c,\alpha\rangle} + (\text{lower order terms of } n) \\
&= \frac{1}{(d-1)!}\left\{ (d-1)! \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle\psi,\psi\rangle}{2\langle\rho_c,\alpha\rangle} \right\} n^{d-1} + (\text{lower order terms of } n)
\end{aligned}$$

From the last formula, we can read off the desired formulas of dimension and degree.

On the other hand, since Y is a $K_{\mathbb{C}}$-orbit through a highest weight vector in $\mathfrak{p}$, $\overline{Y}$ is a highest weight variety (see [**48**]). Then the decomposition of the coordinate ring as a $K_{\mathbb{C}}$-module becomes

$$\mathbb{C}[\overline{Y}] \simeq \bigoplus_{m\ge 0} \tau_{m\psi},$$

with grading given by m. By the same method as above, we conclude that $\deg \overline{Y}$ is equal to $\operatorname{Deg} \pi_{\min}$ which proves that $\mathcal{AC}(\pi_{\min}) = [\overline{Y}]$. □

3. Reductive dual pair

Let W be a real symplectic space of dimension $2N$. We put $\mathcal{G} = Sp(W) = Sp(2N, \mathbb{R})$ and $\widetilde{\mathcal{G}} = Mp(2N, \mathbb{R})$, the metaplectic double cover of $\mathcal{G}$ (see [**47**, § I.2] for example). A pair of reductive subgroups (G_1, G_2) of $\mathcal{G}$ is called a *reductive dual pair* if they are mutually commutant to each other in $\mathcal{G}$ (see [**22**], for example). We denote by $(\widetilde{G_1}, \widetilde{G_2})$ the inverse image of these subgroups under the covering map $\widetilde{\mathcal{G}} \to \mathcal{G}$. Then they are also commutant to each other in $\widetilde{\mathcal{G}}$.

Let us assume that the pair (G_1, G_2) is irreducible (see [**25**, § 4] for definition). Then there are two possibilities.

(I) The pair (G_1, G_2) jointly acts on W. This action is irreducible.

(II) There exists a maximally totally isotropic space U of W, such that $W = U \oplus U^*$ gives the irreducible decomposition with respect to the joint action of the pair.

In the following, *we only treat the dual pair of type* (I), so that we assume that the joint action of $G_1 \times G_2$ on W is irreducible. Then, by the irreducibility, there exist a division algebra D over $\mathbb{R}$ and vector spaces V_1/D and $D\backslash V_2$ over D for which the following two properties hold. First, W is the tensor product of V_1 and V_2 over D :

$$W = V_1 \otimes_D V_2.$$

Second, G_i $(i = 1, 2)$ acts on V_i irreducibly as D-linear transformations. We put

$$2n = \dim_{\mathbb{R}} V_1, \qquad m = \dim_D V_2, \tag{3.12}$$

hence $\dim_{\mathbb{R}} W = 2N = 2nm$. Note that the division algebra is given by $D \simeq \operatorname{End}_{G_1}(V_1) \simeq \operatorname{End}_{G_2}(V_2)$.

Since W carries a symplectic structure (and (G_1, G_2) is a pair in the symplectic group $Sp(W)$), it produces some additional structure on the vector spaces V_1 and V_2. Namely, we have the following.

First, there exists an involution ι of D (possibly trivial). Second, V_i $(i = 1, 2)$ carries a sesqui-linear form $(\ ,\ )_i$ which is invariant under G_i. One of the forms, say $(\ ,\ )_1$, is skew-Hermitian with respect to the involution ι and the other $(\ ,\ )_2$ is Hermitian; and the original symplectic form $\langle\ ,\ \rangle_W$ on W is given by the product of these forms:

$$\langle\ ,\ \rangle_W = \operatorname{Re}(\ ,\ )_1 \otimes_D (\ ,\ )_2.$$

Moreover, the group G_i is the full isometry group with respect to $(\ ,\ )_i$. In the following, we always assume that $(\ ,\ )_1$ is skew-Hermitian, and $(\ ,\ )_2$ is Hermitian.

Here is a table (Table 1) of such pairs borrowed from [**25**, Table 4.1].

In this paper, we only treat the case where one of the pair, say G_2, is compact. In fact, we have the following explicit cases in Table 2 in mind. However, we try to keep general situation whenever possible. In any case, G_2 *is always assumed to be compact.*

Let us specify an explicit embedding of (G_1, G_2) into $\mathcal{G} = Sp(2nm, \mathbb{R})$. Although our arguments below are fairly general, sometimes it is convenient to use a concrete realization. In each of three cases, we will give a symplectic vector space $\mathbb{R}^{2nm}$

TABLE 1. Reductive dual pairs of type (I).

(D,ι)	$\mathcal{G}$	(G_1,G_2)	
$(\mathbb{R},\mathbf{1})$	$Sp(2nm,\mathbb{R})$	$(Sp(2n,\mathbb{R}),O(p,q))$	$m=p+q$
$(\mathbb{C},\mathbf{1})$	$Sp(4nm,\mathbb{R})$	$(Sp(2n,\mathbb{C}),O(m,\mathbb{C}))$	
$(\mathbb{C},^{-})$	$Sp(2nm,\mathbb{R})$	$(U(p,q),U(r,s))$	$n=p+q, m=r+s$
$(\mathbb{H},^{-})$	$Sp(2nm,\mathbb{R})$	$(O^*(2p),Sp(r,s))$	$n=2p, m=r+s$

TABLE 2. Reductive dual pairs (G_1,G_2) with G_2 being compact.

	(D,ι)	$\mathcal{G}$	(G_1,G_2)	
Case (Sp,O)	$(\mathbb{R},\mathbf{1})$	$Sp(2nm,\mathbb{R})$	$(Sp(2n,\mathbb{R}),O(m))$	
Case (U,U)	$(\mathbb{C},^{-})$	$Sp(2nm,\mathbb{R})$	$(U(p,q),U(m))$	$n=p+q$
Case (O^*,Sp)	$(\mathbb{H},^{-})$	$Sp(2nm,\mathbb{R})$	$(O^*(2p),Sp(2m))$	$n=2p$

endowed with an explicit symplectic form in terms of invariant bilinear forms of V_1 and V_2. This will determine the group $\mathcal{G}=Sp(2nm,\mathbb{R})$.

Case (Sp,O)**.** Let $\mathbb{R}^{2n}$ be a symplectic vector space with a symplectic form

$$(u,v)_1={}^t u J_n v \quad (u,v\in\mathbb{R}^{2n}), \qquad J_n=\begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}, \tag{3.13}$$

and consider $G_1=Sp(2n,\mathbb{R})$ as the isometry group of $(\mathbb{R}^{2n},(\ ,\ )_1)$. For $G_2=O(m)$, we take the standard Euclidean bilinear form $(u,v)_2={}^t uv \quad (u,v\in\mathbb{R}^m)$, and consider $O(m)=O(\mathbb{R}^m,(\ ,\ )_2)$. Then the tensor product $W=\mathbb{R}^{2n}\otimes_{\mathbb{R}}\mathbb{R}^m$ with a symplectic form

$$\langle\ ,\ \rangle_W=(\ ,\ )_1\otimes_{\mathbb{R}}(\ ,\ )_2$$

gives the embedding $(G_1,G_2)\hookrightarrow\mathcal{G}=Sp(W,\langle\ ,\ \rangle_W)$.

Let us see this embedding infinitesimally. So, first consider $\mathfrak{sp}(2n,\mathbb{R})$:

$$\begin{aligned}\mathfrak{sp}(2n,\mathbb{R}) &= \left\{Z\in\mathfrak{gl}(2n,\mathbb{R}) \mid {}^t Z J_n + J_n Z=0\right\} \\ &= \left\{\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -{}^t X_{11}\end{pmatrix} \;\middle|\; \begin{array}{l} X_{11}\in\mathfrak{gl}(n,\mathbb{R}), \\ X_{12},X_{21}\in\mathrm{Sym}\,(n,\mathbb{R})\end{array}\right\}.\end{aligned} \tag{3.14}$$

Then it is embedded into larger $\mathfrak{sp}(2nm,\mathbb{R})$ as

$$\mathfrak{sp}(2n,\mathbb{R}) \ni \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -{}^tX_{11} \end{pmatrix} \longmapsto \begin{pmatrix} X_{11}^{\oplus m} & X_{12}^{\oplus m} \\ X_{21}^{\oplus m} & -{}^tX_{11}{}^{\oplus m} \end{pmatrix} \in \mathfrak{sp}(2nm,\mathbb{R}), \quad (3.15)$$

where

$$X^{\oplus m} = \operatorname{diag}(X, X, \ldots, X) \quad (m\text{-times}).$$

Similarly, $\mathfrak{o}(m,\mathbb{R}) = \operatorname{Alt}(m,\mathbb{R})$ is embedded into $\mathfrak{sp}(2nm,\mathbb{R})$ as

$$\mathfrak{o}(m,\mathbb{R}) \ni X \longmapsto \begin{pmatrix} X * 1_n & 0 \\ 0 & X * 1_n \end{pmatrix},$$

where

$$X * A = \begin{pmatrix} x_{11}A & x_{12}A & \cdots & x_{1m}A \\ x_{21}A & x_{22}A & \cdots & x_{2m}A \\ \vdots & \vdots & & \vdots \\ x_{m1}A & x_{m2}A & \cdots & x_{mm}A \end{pmatrix}. \quad (3.16)$$

Case (U, U)**.** Consider an indefinite Hermitian form $(\ ,\ )_1$ on $\mathbb{C}^n$ of signature (p,q) $(n = p+q)$:

$$(u,v)_1 = {}^t\overline{u} I_{p,q} v \quad (u, v \in \mathbb{C}^n), \qquad I_{p,q} = \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix}. \quad (3.17)$$

Then, $G_1 = U(p,q)$ is the full isometry group of $(\mathbb{C}^n, (\ ,\ )_1)$. Also we take a definite Hermitian form $(\ ,\ )_2$ on $\mathbb{C}^m$ as $(u,v)_2 = {}^t\overline{u}v$ $(u, v \in \mathbb{C}^m)$. This determines the unitary group $G_2 = U(m)$. Then the tensor product $W = \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m$ naturally inherits a Hermitian form $(\ ,\ )_1 \otimes_{\mathbb{C}} (\ ,\ )_2$. We make use of its imaginary part to define a symplectic form on $W \simeq \mathbb{R}^{2nm}$:

$$\langle\ ,\ \rangle_W = \operatorname{Re}\left(\sqrt{-1}\,(\ ,\ )_1 \otimes_{\mathbb{C}} (\ ,\ )_2\right).$$

The form $\langle\ ,\ \rangle_W$ is clearly non-degenerate and it defines the isometry group $\mathcal{G} = Sp(W, \langle\ ,\ \rangle_W) \simeq Sp(2nm,\mathbb{R})$.

Under our explicit realization of $U(p,q)$, its Lie algebra is given as

$$\begin{aligned} \mathfrak{u}(p,q) &= \left\{ Z \in \mathfrak{gl}(p+q,\mathbb{C}) \mid {}^t\overline{Z} I_{p,q} + I_{p,q} Z = 0 \right\} \\ &= \left\{ Z = \begin{pmatrix} Z_{11} & Z_{12} \\ {}^t\overline{Z_{12}} & Z_{22} \end{pmatrix} \;\middle|\; \begin{matrix} Z_{11} \in \text{skew-Her}\,(p,\mathbb{C}) \\ Z_{22} \in \text{skew-Her}\,(q,\mathbb{C}) \\ Z_{12} \in M(p,q,\mathbb{C}) \end{matrix} \right\}. \end{aligned} \quad (3.18)$$

Let us write $Z = X + \sqrt{-1}\,Y$ with $X, Y \in M(n,\mathbb{R})$. Then an explicit embedding into $\mathfrak{sp}(2n,\mathbb{R})$ is given by

$$\mathfrak{u}(p,q) \ni X + \sqrt{-1}\,Y \mapsto \begin{pmatrix} X & -YI_{p,q} \\ I_{p,q}Y & I_{p,q}XI_{p,q} \end{pmatrix} \in \mathfrak{sp}(2n,\mathbb{R}). \quad (3.19)$$

Now the above embedding composed by the embedding (3.15) will give the desired realization of $\mathfrak{u}(p,q)$ in $\mathfrak{sp}(2nm,\mathbb{R})$.

On the other hand, the compact companion $\mathfrak{u}(m)$ is embedded into $\mathfrak{sp}(2nm,\mathbb{R})$ as

$$\begin{aligned}\mathfrak{u}(m) &= \mathrm{Alt}\,(m,\mathbb{R}) + \sqrt{-1}\ \mathrm{Sym}\,(m,\mathbb{R}) \ni X + \sqrt{-1}\,Y \\ &\longmapsto \begin{pmatrix} X * 1_n & -Y * I_{p,q} \\ Y * I_{p,q} & X * 1_n \end{pmatrix} \in \mathfrak{sp}(2nm,\mathbb{R}). \end{aligned} \tag{3.20}$$

Case (O^*, Sp)**.** Let $O(2p,\mathbb{C})$ be the complex orthogonal group with respect to the following bilinear form

$$(u,v) = {}^t u S_p v \quad (u,v \in \mathbb{C}^{2p}), \qquad S_p = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}.$$

We realize $G_1 = O^*(2p)$ as a subgroup of $O(2p,\mathbb{C})$, namely,

$$O^*(2p) = O(2p,\mathbb{C}) \cap U(p,p) \subset M(2p,\mathbb{C}), \tag{3.21}$$

where $U(p,p)$ is realized in the same way as Case (U,U). Similarly, we realize $G_2 = Sp(2m)$ as a compact subgroup of $Sp(2m,\mathbb{C})$:

$$Sp(2m) = Sp(2m,\mathbb{C}) \cap U(2m) \subset M(2m,\mathbb{C}).$$

First we describe embedding of $\mathfrak{o}^*(2p)$ into $\mathfrak{sp}(2n,\mathbb{R})$ $(n = 2p)$. Our realization of $O^*(2p)$ gives its Lie algebra as

$$\begin{aligned}\mathfrak{o}^*(2p) &= \{Z \in \mathfrak{gl}(2p,\mathbb{C}) \mid {}^t\overline{Z} I_{p,p} + I_{p,p} Z = 0, \quad {}^t Z S_p + S_p Z = 0\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ \overline{Y} & \overline{X} \end{pmatrix} \mid X \in \text{skew-Her}\,(p,\mathbb{C}),\ Y \in \mathrm{Alt}\,(p,\mathbb{C}) \right\},\end{aligned}$$

where $I_{p,p}$ is given by (3.17).

It is subtle to describe a symplectic form of the larger $Sp(2nm,\mathbb{R})$ $(n = 2p)$ in terms of the original (skew-)Hermitian forms over $\mathbb{H}$ which define $O^*(2p)$ and $Sp(2m)$ as the full isometry groups. Instead, we give here only an explicit embedding of $O^*(2p)$ infinitesimally. Let us write $X = X_1 + \sqrt{-1}\ X_2$ and $Y = Y_1 + \sqrt{-1}\ Y_2$ with real matrices X_i, Y_i $(i = 1,2)$. Then, the infinitesimal embedding of $\mathfrak{o}^*(2p)$ into $\mathfrak{sp}(2n,\mathbb{R})$ is given by

$$\mathfrak{o}^*(2p) \ni \begin{pmatrix} X & -Y \\ \overline{Y} & \overline{X} \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} X_1 & -Y_1 & -X_2 & -Y_2 \\ Y_1 & X_1 & Y_2 & -X_2 \\ \hline X_2 & -Y_2 & X_1 & Y_1 \\ Y_2 & X_2 & -Y_1 & X_1 \end{array}\right) \in \mathfrak{sp}(2n,\mathbb{R}). \tag{3.22}$$

This embedding is compatible with the embedding given in Case (U,U), i.e., we have a sequence of subgroups

$$O^*(2p) \hookrightarrow U(p,p) \hookrightarrow Sp(2n,\mathbb{R}).$$

The embedding into the larger $\mathfrak{sp}(2nm,\mathbb{R})$ is given by the composition of (3.15) and (3.22).

Let us see the embedding of the compact companion $\mathfrak{sp}(2m)$. Its Lie algebra becomes

$$\begin{aligned}\mathfrak{sp}(2m) &= \{Z \in \mathfrak{gl}(2m, \mathbb{C}) \mid {}^t\overline{Z} + Z = 0, \quad {}^tZJ_m + J_mZ = 0\} \\ &= \left\{Z = \begin{pmatrix} X & -Y \\ \overline{Y} & \overline{X} \end{pmatrix} \mid X \in \text{skew-Her}\,(m, \mathbb{C}),\ Y \in \text{Sym}\,(m, \mathbb{C})\right\}.\end{aligned}$$

If we denote $Z = A + \sqrt{-1}\, B$ with real matrices A and B, then the embedding is given by

$$\mathfrak{sp}(2m) \ni Z = A + \sqrt{-1}\, B \longmapsto \begin{pmatrix} A * 1_p & -B * 1_p \\ B * 1_p & A * 1_p \end{pmatrix} \in \mathfrak{sp}(2nm, \mathbb{R}).$$

4. Fock realization of Weil representation

Let ω be the Weil representation of $Mp(2n, \mathbb{R})$, the metaplectic double cover of $Sp(2n, \mathbb{R})$. Weil representation plays central roles in many fields, and a large amount of results are known. For example, see [**24**], [**25**], [**28**], [**47**], [**40**], etc. We introduce here, among all, explicit realization of Harish-Chandra module of ω on a polynomial ring (e.g., see [**25**] and [**9**]). It is called *Fock model*.

For the time being, we write $G = Sp(2n, \mathbb{R})$ and $\widetilde{G} = Mp(2n, \mathbb{R})$. Since we only consider Harish-Chandra modules, in fact we do not need entire $Mp(2n, \mathbb{R})$ but only its complexified Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and a maximal compact subgroup $\widetilde{K} = U(n)^{\sim}$. We fix a maximal compact subgroup $K \simeq U(n)$ in $Sp(2n, \mathbb{R})$ as follows. Put

$$Sp(2n, \mathbb{R}) = \left\{ g \in GL(2n, \mathbb{R}) \mid {}^t g J_n g = J_n \right\}, \qquad J_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}. \tag{4.23}$$

Then K is given as

$$K = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in M(n, \mathbb{R}), \quad a + ib \in U(n) \right\}. \tag{4.24}$$

We identify K and $U(n)$ as above and sometimes we will write $a + ib \in K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the corresponding Cartan decomposition, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its complexification.

Let E_{ij} be the matrix unit, and put

$$F_{ij} := E_{ij} - E_{ji}, \qquad G_{ij} := E_{ij} + E_{ji}.$$

Then it is easy to see that a basis of $\mathfrak{k}$ is given by

$$A_{ij} := \begin{pmatrix} F_{ij} & O_n \\ O_n & F_{ij} \end{pmatrix} \quad (1 \le i < j \le n), \qquad B_{ij} := \begin{pmatrix} O_n & -G_{ij} \\ G_{ij} & O_n \end{pmatrix} \quad (1 \le i \le j \le n), \tag{4.25}$$

and that of $\mathfrak{p}$ is given by

$$C_{ij} := \begin{pmatrix} G_{ij} & O_n \\ O_n & -G_{ij} \end{pmatrix} \quad (1 \le i \le j \le n), \qquad D_{ij} := \begin{pmatrix} O_n & G_{ij} \\ G_{ij} & O_n \end{pmatrix} \quad (1 \le i \le j \le n). \tag{4.26}$$

The representation space of ω in Fock model is a polynomial ring in n variables. Here we only give the explicit action of each basis element on the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$:

$$\begin{aligned} \omega(A_{ij}) &= x_i \partial_{x_j} - x_j \partial_{x_i}, & \omega(B_{ij}) &= \sqrt{-1}\,(x_i \partial_{x_j} + \partial_{x_i} x_j), \\ \omega(C_{ij}) &= 2\partial_{x_i}\partial_{x_j} - \tfrac{1}{2} x_i x_j, & \omega(D_{ij}) &= -\sqrt{-1}\,(2\partial_{x_i}\partial_{x_j} + \tfrac{1}{2} x_i x_j). \end{aligned} \tag{4.27}$$

The action of $\widetilde{K}$ on $\mathbb{C}[x_1, \dots, x_n]$ is the symmetric tensor product of the natural representation of $U(n)$ on $\mathbb{C}^n$ tensored by $\det^{1/2}$, which requires the double cover $\widetilde{K}$.

Put $H_i = -\sqrt{-1}\, B_{ii}/2$ and let

$$\mathfrak{t} = \left\{ \sum_{i=1}^{n} t_i H_i \mid t_i \in \mathbb{C} \right\}$$

be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. We define $\varepsilon_j \in \mathfrak{t}^*$ as $\varepsilon_j(H_i) = \delta_{ij}$. Then the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ is given by

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \leq i \leq j \leq n\},$$

where $\varepsilon_i - \varepsilon_j$ is a compact root while $\pm(\varepsilon_i + \varepsilon_j)$ is non-compact. We take a positive system in the standard way :

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq n\}.$$

Then root vectors X_α $(\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}))$ and its action on Weil representation are given as

$$H_i = -\frac{\sqrt{-1}}{2} B_{ii}, \qquad \omega(H_i) = x_i \partial_{x_i} + \frac{1}{2}, \tag{4.28}$$

$$X_{\varepsilon_i - \varepsilon_j} = \frac{1}{2}(A_{ij} - \sqrt{-1}\, B_{ij}) \quad (i \neq j), \qquad \omega(X_{\varepsilon_i - \varepsilon_j}) = x_i \partial_{x_j}, \tag{4.29}$$

$$X_{\varepsilon_i + \varepsilon_j} = -\frac{1}{2}(C_{ij} - \sqrt{-1}\, D_{ij}), \qquad \omega(X_{\varepsilon_i + \varepsilon_j}) = \frac{1}{2} x_i x_j, \tag{4.30}$$

$$X_{-\varepsilon_i - \varepsilon_j} = \frac{1}{2}(C_{ij} + \sqrt{-1}\, D_{ij}), \qquad \omega(X_{-\varepsilon_i - \varepsilon_j}) = 2\partial_{x_i} \partial_{x_j}. \tag{4.31}$$

Note that

$$\mathfrak{k} \simeq \mathfrak{gl}(n, \mathbb{C}) \ni E_{ij} \leftrightarrow \begin{cases} X_{\varepsilon_i - \varepsilon_j} = x_i \partial_{x_j} & (i \neq j), \\ H_i = x_i \partial_{x_i} + \dfrac{1}{2} & (i = j). \end{cases} \tag{4.32}$$

We write

$$\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i, j \leq n\}, \qquad \Delta_n = \Delta_n^+ \sqcup (-\Delta_n^+),$$

the set of non-compact roots, and

$$\Delta_k^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}, \qquad \Delta_k = \Delta_k^+ \sqcup (-\Delta_k^+),$$

the set of compact roots. Then $\mathfrak{p}$ decomposes up into two K-irreducible components $\mathfrak{p}^\pm$ given by

$$\mathfrak{p}^\pm = \sum_{\pm\alpha \in \Delta_n^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the roots space corresponding to α. Note that $\omega(\mathfrak{p}^-)$ is realized as differential operators of degree two, and that $\omega(\mathfrak{p}^+)$ is the multiplication by homogeneous polynomials of degree two. So $\mathfrak{p}^\pm$ increases/decreases the degree of the representation space $\mathbb{C}[x_1, x_2, \ldots, x_n]$ by 2, while $\omega(\mathfrak{k})$ keeps the degree stable.

5. Unitary lowest weight representations

Let Ω be the Weil representation of $\widetilde{\mathcal{G}} = Mp(2N, \mathbb{R})$ $(N = nm)$ and consider reductive dual pair (G_1, G_2) of compact type in $\mathcal{G} = Sp(2N, \mathbb{R})$. In the following, we often write $G = G_1$ without the subscription 1. In fact, our main concern is on the irreducible infinite dimensional representations of $G = G_1$ which appear in the restriction of Weil representation Ω. Moreover, we assume that G_2 is contained in the specified maximal compact subgroup $\mathcal{K} \simeq U(N)$ of $\mathcal{G}$ given in the former section (cf. (4.24)). Each of our three cases (and their realization) clearly satisfies this condition.

For a subgroup $H \subset \mathcal{G}$, we denote by $\widetilde{H}$ the inverse image of H in $\widetilde{\mathcal{G}}$ of the covering map $\widetilde{\mathcal{G}} \to \mathcal{G}$, and call it the metaplectic cover of H by abuse of terminology. Since the metaplectic covers $\widetilde{G_1}$ and $\widetilde{G_2}$ commute with each other, we have a natural projection $\widetilde{G_1} \times \widetilde{G_2} \to \widetilde{G_1} \cdot \widetilde{G_2}$ (product in $\widetilde{\mathcal{G}}$). By this projection, we consider the restriction $\Omega\big|_{\widetilde{G_1}\cdot\widetilde{G_2}}$ as a representation of $\widetilde{G_1} \times \widetilde{G_2}$. Then we have a discrete and multiplicity free decomposition

$$\Omega \simeq \sum_{\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G_2})}^{\oplus} L(\widetilde{\sigma} \otimes \chi^{-1}) \boxtimes \widetilde{\sigma} \tag{5.33}$$

as a representation of $\widetilde{G_1} \times \widetilde{G_2}$. Here we denote an irreducible representation of $\widetilde{G} = \widetilde{G_1}$ corresponding to $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G_2})$ by $L(\widetilde{\sigma} \otimes \chi^{-1})$, where $\chi \in \mathrm{Irr}(\widetilde{G_2})$ is the unique one-dimensional character which appears in $\Omega\big|_{\widetilde{G_2}}$ (cf. Theorem 4.3 in [**25**]).

To be more specific, we argue like this. The representation space of Ω is realized on a polynomial ring of $N = nm$ variables. We consider it as the polynomial ring on the *dual* space of $n \times m$ matrices $M_{n,m}$ over $\mathbb{C}$. Since the one-dimensional space of constant polynomials in $\mathbb{C}[M_{n,m}^*]$ is preserved by the action of $\widetilde{\mathcal{K}}$, it is also preserved by $\widetilde{G_2}$ because of our assumption. Hence it gives the one-dimensional character and we denote it by $\chi \in \mathrm{Irr}(\widetilde{G_2})$.

The representation $L(\widetilde{\sigma} \otimes \chi^{-1})$ is possibly zero, and if it is not zero, then the representation $\sigma = \widetilde{\sigma} \otimes \chi^{-1}$ factors through to the representation of G_2. Therefore we write $L(\sigma) = L(\widetilde{\sigma} \otimes \chi^{-1})$ for $\sigma \in \mathrm{Irr}(G_2)$. The decomposition (5.33) can be rewritten as

$$\Omega \simeq \sum_{\sigma \in \mathrm{Irr}(G_2)}^{\oplus} L(\sigma) \boxtimes (\sigma \otimes \chi). \tag{5.34}$$

In the following, as explained above, we always twist the representation $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G_2})$ by χ, and consider $\sigma = \widetilde{\sigma} \otimes \chi^{-1}$ as the representation of G_2. This twist might be sometimes misleading, but it reduces considerable amount of *untwisting*. For example, under this convention, we have $L(\mathbf{1}_{G_2}) \neq 0$, where $\mathbf{1}_{G_2}$ denotes the trivial representation of G_2. This representation turns out to be strongly related to geometric properties of nilpotent orbits.

It is known that $L(\sigma)$ $(\sigma \in \mathrm{Irr}(G_2))$ is an irreducible unitary lowest weight module of $\widetilde{G}$, if it is not zero (cf. Theorem 4.4 in [**25**]). Every irreducible unitary lowest weight module of $\widetilde{G}$ arises in this manner if $G = Sp(2n,\mathbb{R})$ or $U(p,q)$ and the compact companion G_2 moves all the possible rank. If G is $O^*(2p)$, there are other unitary lowest weight modules which can not be obtained in this manner ([**9**], [**11**]).

In our cases, the compact subgroup G_2 naturally acts on its defining vector space V_2 keeping the non-degenerate Hermitian form $(\ ,\ )_2$ invariant (see § 3). Put $2n = \dim_{\mathbb{R}} V_1$ and $m = \dim_D V_2$ as in (3.12). Then we can realize G in a smaller symplectic group: $G \hookrightarrow Sp(2n,\mathbb{R})$, putting $m = 1$.

Let us denote the Weil representation of the smaller group $Mp(2n,\mathbb{R})$ by ω. Then, it is easy to see that $\Omega \simeq \omega^{\otimes m}$ as a representation of $Mp(2n,\mathbb{R})$, and the Harish-Chandra $(\mathfrak{g},K)$-module of the Weil representation Ω (resp. ω) is realized on the polynomial ring $\mathbb{C}[M_{n,m}^*] \simeq \otimes^m \mathbb{C}[(\mathbb{C}^n)^*]$ over $n\times m$ matrices (resp. the polynomial ring $\mathbb{C}[(\mathbb{C}^n)^*]$). Note that we take a contragredient representation of $M_{n,m}$ rather than $M_{n,m}$ itself.

Let $K \subset G$ be a maximal compact subgroup of G which lives in $U(n) \subset Sp(2n,\mathbb{R})$, where $U(n)$ is a maximal compact subgroup of $Sp(2n,\mathbb{R})$ (cf. (4.24)). We will explain briefly how we get K-type decomposition of $L(\sigma)$ for each $\sigma \in \mathrm{Irr}(G_2)$. Note that the product $K\cdot G_2$ is compact and that it is contained in the maximal compact subgroup $\mathcal{K} \simeq U(nm)$ of $\mathcal{G} = Sp(2nm,\mathbb{R})$. It is well-known that $\mathcal{K}$-types of Ω can be described as

$$\Omega\big|_{\widetilde{\mathcal{K}}} = \sum_{k=0}^{\infty}{}^{\oplus}\, \tau(k\psi + 1/2\,\mathbb{I}), \quad \mathbb{I} = (1,1,\dots,1), \tag{5.35}$$

where ψ is the highest weight of the natural (or defining) representation of $\mathcal{K} \simeq U(nm)$ on $\mathbb{C}^{nm}$ and $\tau(\lambda)$ is an irreducible finite-dimensional representation of $\mathcal{K}$ with highest weight λ. Note that the representation space of $\tau(k\psi + 1/2\,\mathbb{I})$ coincides with the space of homogeneous polynomials of degree k. Decompose $\tau(k\psi + 1/2\,\mathbb{I})$ by the joint action of $\widetilde{K}\times\widetilde{G_2}$:

$$\tau(k\psi + 1/2\,\mathbb{I})\big|_{\widetilde{K}\times\widetilde{G_2}} = \sum_{\tau_1\in\mathrm{Irr}(\widetilde{K}),\ \sigma\in\mathrm{Irr}(G_2)}^{\oplus} m_k(\tau_1,\sigma)\,\tau_1 \boxtimes \widetilde{\sigma} \qquad (\widetilde{\sigma} = \sigma\otimes\chi).$$

Note that we again use the projection map $\widetilde{K}\times\widetilde{G_2} \to \widetilde{K}\cdot\widetilde{G_2} \subset \widetilde{\mathcal{K}}$ here. In particular, the one-dimensional space $\tau(1/2\,\mathbb{I})$ is decomposed as

$$\tau(1/2\,\mathbb{I})\big|_{\widetilde{K}\times\widetilde{G_2}} = \chi_1 \boxtimes \chi.$$

In other words, the multiplicity for $k = 0$ has the property

$$m_0(\tau_1,\sigma) = \begin{cases} 1 & \tau_1 = \chi_1, \sigma = \mathbf{1}_{G_2} \\ 0 & \text{otherwise.} \end{cases}$$

The explicit form of χ and χ_1 is given in Section 7 after we fix the embedding $K \subset \mathcal{K}$.

Since $L(\sigma)$ consists of the space of multiplicities of $\widetilde{\sigma}$ in Ω, we get K-type decomposition of $L(\sigma)$ as

$$L(\sigma)\big|_{\widetilde{K}} = \sum_{\tau_1\in\mathrm{Irr}(\widetilde{K})}^{\oplus} \sum_{k=0}^{\infty} m_k(\tau_1,\sigma)\ \tau_1, \tag{5.36}$$

where the sum $\sum_{k=0}^{\infty} m_k(\tau_1, \sigma)$ is necessarily finite.

Let k_σ be the lowest possible degree of $\widetilde{\sigma}$-isotypic component of $\mathbb{C}[M_{n,m}^*]$. We define the Poincaré series of $L(\sigma)$ in terms of the multiplicity $m_k(\tau_1, \sigma)$ as

$$P(L(\sigma); t^2) = t^{-k_\sigma} \sum_{k=0}^{\infty} \sum_{\tau_1 \in \mathrm{Irr}(\widetilde{K})} m_k(\tau_1, \sigma)\ \dim \tau_1\ t^k. \tag{5.37}$$

Note that the action of $\mathfrak{p}^+$ increases the degree k by two (cf. (4.30)), so we write $P(L(\sigma); t^2)$ instead of $P(L(\sigma); t)$. We denote the center of $\widetilde{\mathcal{K}}$ by $Z(\widetilde{\mathcal{K}})$. We know that $Z(\widetilde{\mathcal{K}})$ is isomorphic to $U(1)$ and there exists an element H in the Lie algebra of $Z(\widetilde{\mathcal{K}})$ such that $\Omega(H)$ acts on the space of homogeneous polynomials of degree k by $k + nm/2$. Indeed, $H = \sum_{i=1}^{nm} H_i$ with the notation (4.32). The operator $\Omega(H)$ is semisimple, and the decomposition into the H-isotypic components is given by (5.35). Moreover, the natural embedding $\widetilde{K} \subset \widetilde{\mathcal{K}}$ induces an isomorphism between the Lie algebra of $Z(\widetilde{K})$ and that of $Z(\widetilde{\mathcal{K}})$. We denote the element in the Lie algebra of $Z(\widetilde{K})$ corresponding to H by H'. Then the formal character of $L(\sigma)$ on the compact Cartan subgroup restricted to the center of $\widetilde{K}$ can be expressed by the Poincaré series:

$$\mathrm{trace}_{\,L(\sigma)} t^{H'} = t^{k_\sigma + nm/2} P(L(\sigma); t^2). \tag{5.38}$$

To get explicit multiplicity formulas, we are involved in case-by-case analysis.

6. Description of K-types of the lowest weight modules

Assume that the pair (G_1, G_2) is in the *stable range* where G_2 is the smaller member. This means that $m \leq \mathbb{R}\text{-rank}\, G_1$, where $m = \dim_D V_2$ (cf. §3). Take $\sigma \in \mathrm{Irr}(G_2)$ for which $L(\sigma)$ is not zero, and put $\widetilde{\sigma} = \sigma \otimes \chi$ as above. Let us describe K-type decomposition of $L(\sigma)$ in each explicit cases.

6.1. Case (Sp, O). — Assume that $m \leq n = \mathbb{R}\text{-rank}\, Sp(2n, \mathbb{R})$. This means the pair $(Sp(2n, \mathbb{R}), O(m))$ is in the stable range. As before, we shall write $G = G_1 = Sp(2n, \mathbb{R})$ and $\mathcal{G} = Sp(2nm, \mathbb{R})$.

Let $K = U(n)$ be a maximal compact subgroup of $G = Sp(2n, \mathbb{R})$ which is realized in the standard way (cf. (4.24)). Let $\mathcal{K} = U(n \times m) \subset \mathcal{G}$ act on $M_{n,m} = M(n, m, \mathbb{C})$ naturally as unitary transformation group. The product group $U(n) \times O(m)$ acts on $M_{n,m}$ naturally as

$$(k, h)X = kX\,{}^t h \qquad ((k, h) \in U(n) \times O(m),\ X \in M_{n,m}). \tag{6.39}$$

Since the action is also unitary, it induces a map $U(n) \times O(m) \to U(nm) = \mathcal{K}$. The image of the above map coincides with $K \cdot G_2$. Note that the kernel of the map is $\{(\pm 1_n, \pm 1_m) \in U(n) \times O(m)\}$.

The metaplectic cover $\widetilde{\mathcal{K}}$ acts on $M_{n,m}$ as the composition of the projection $\widetilde{\mathcal{K}} \to \mathcal{K}$ and the natural action of the unitary group $\mathcal{K} = U(nm)$. This action induces the representation of $\widetilde{\mathcal{K}}$ on the polynomial ring $\mathbb{C}[M_{n,m}^*]$, which is isomorphic to the symmetric tensor of $M_{n,m}$. By the formula (4.32), we conclude that the action of $\widetilde{\mathcal{K}}$ on $\mathbb{C}[M_{n,m}^*]$ via Weil representation Ω is twisted by $\det^{1/2}$. We shall denote this representation by $\mathbb{C}[M_{n,m}^*] \otimes \det^{1/2}$. Therefore $\widetilde{K}$ acts on $\mathbb{C}[M_{n,m}^*]$ as $\mathbb{C}[M_{n,m}^*] \otimes \det^{m/2}$ and $\widetilde{G_2}$ acts as $\mathbb{C}[M_{n,m}^*] \otimes \det^{n/2}$. So the one-dimensional representation χ_1 of $\widetilde{K}$ coincides with $\det^{m/2}$, and the one-dimensional representation $\chi \in \mathrm{Irr}(\widetilde{G_2})$ coincides with $\det^{n/2}$. However, we should be more precise about χ because $G_2 = O(m)$ is not connected.

The metaplectic cover $\widetilde{\mathcal{K}}$ has a realization

$$\widetilde{\mathcal{K}} = \{(k, z) \in \mathcal{K} \times \mathbb{C}^\times \mid \det k = z^2\}$$

and the representation $\det^{1/2}$ of $\widetilde{\mathcal{K}}$ is given by the map $(k, z) \mapsto z$. Then the subgroup $\widetilde{G_2}$ is realized as

$$\widetilde{G_2} = \{(k, z) \in G_2 \times \mathbb{C}^\times \mid \det^n k = z^2\}$$

and its character $\chi = \det^{n/2}$ is given by $\chi(k, z) = z$. The identity component of $\widetilde{G_2}$ is

$$\big(\widetilde{G_2}\big)_0 = \{(k, z) \in G_2 \times \mathbb{C}^\times \mid \det k = 1, z = 1\} \simeq SO(m).$$

The map $(k, z) \mapsto (\det k, z)$ induces the isomorphism of the component group $\widetilde{G_2}/\big(\widetilde{G_2}\big)_0$ onto the group

$$A(G_2) = \{(t, z) \in \mathbb{Z}_2 \times \mathbb{C}^\times \mid t^n = z^2\}$$

of order four. Since the one-dimensional character χ is trivial on the identity component $(\widetilde{G_2})_0$, it induces the character of the component group $A(G_2)$. We denote it by the same letter χ, then $\chi(t,z) = z$. First, we consider the case where n is odd. Then $A(G_2) = \{(\zeta^2,\zeta) \mid \zeta = \pm 1, \pm\sqrt{-1}\ \} \cong \mathbb{Z}_4$. If we define $\varepsilon = ((\mathrm{diag}\,(1_{m-1},-1),\sqrt{-1}\ ) \in \widetilde{G_2}$, then $\varepsilon \in A(G_2)$ generates the component group $\mathbb{Z}_4$. Then we see that

$$\widetilde{G_2} \simeq SO(m) \rtimes \mathbb{Z}_4 \qquad \text{if } n \text{ is odd.}$$

The character χ satisfies $\chi(\varepsilon) = \sqrt{-1}$, which determines the character χ of $\mathbb{Z}_4$. Second, let us consider the case where n is even. In this case,

$$\widetilde{G_2} = G_2 \times \mathbb{Z}_2 \qquad \text{if } n \text{ is even.} \tag{6.40}$$

The character χ is trivial on $G_2 = O(m)$ and is non-trivial on $\mathbb{Z}_2$.

By the argument in §5, we will get K-type decomposition of $L(\sigma)$ if we know the explicit decomposition of $\mathbb{C}[M_{n,m}^*] \otimes \det{}^{1/2}$ as $\widetilde{K} \times \widetilde{G_2}$-module. We first consider the space $\mathbb{C}[M_{n,m}^*]$ as the usual symmetric tensor of the natural representation of the unitary group $\mathcal{K} = U(n \times m)$, then afterwards we will twist it by $\det^{1/2}$ to fit it to the Weil representation Ω.

We extend the $U(n) \times O(m)$-action (6.39) on $M_{n,m}$ naturally to the $U(n) \times U(m)$-action. It is well known (cf. [**23**]) that, as $U(n) \times U(m)$-module, $\mathbb{C}[M_{n,m}^*]$ decomposes as

$$\mathbb{C}[M_{n,m}^*]\big|_{U(n)\times U(m)} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} \tau_\lambda^{(n)} \boxtimes \tau_\lambda^{(m)},$$

where $\mathcal{P}_m$ denotes the set of all partitions of length less than or equal to m. We make use of this decomposition. We identify K with $U(n)$ above, and consider $G_2 = O(m)$ in $U(m)$ in the standard manner, i.e., $O(m) = U(m) \cap GL(m,\mathbb{R})$. Decompose $\tau_\lambda^{(m)} \in \mathrm{Irr}(U(m))$ as $O(m)$-module:

$$\tau_\lambda^{(m)}\big|_{O(m)} \simeq \sum_{\sigma\in\mathrm{Irr}(O(m))}^{\oplus} m(\lambda,\sigma)\,\sigma. \tag{6.41}$$

Then we have a joint decomposition

$$\mathbb{C}[M_{n,m}^*]\big|_{U(n)\times O(m)} = \sum_{\sigma\in\mathrm{Irr}(O(m))}^{\oplus} \left\{ \sum_{\lambda\in\mathcal{P}_m}^{\oplus} m(\lambda,\sigma)\,\tau_\lambda^{(n)} \right\} \boxtimes \sigma.$$

So we completely know σ-isotypic component of $\mathbb{C}[M_{n,m}^*]$ in terms of the multiplicity $m(\lambda,\sigma)$. Twist of this representation by $\det^{1/2}$ causes the twist by $\det^{m/2} \boxtimes \chi$ as a representation of $\widetilde{K} \times \widetilde{G_2}$. Therefore $L(\sigma)\big|_{\widetilde{K}}$ decomposes as

$$L(\sigma)\big|_{U(n)^\sim} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} m(\lambda,\sigma)\,\tau_\lambda^{(n)} \otimes \det{}^{m/2}. \tag{6.42}$$

This formula describes the multiplicities of K-types of $L(\sigma)$ in the case of Case (Sp,O).

To describe the lowest weight and the lowest K-type of $L(\sigma)$, we give a classification of $\operatorname{Irr}(O(m))$ briefly. For more detailed discussion, see [**23**, § 3.6] for example. Let $\sigma(\mu)$ be an irreducible representation of $SO(m)$ with highest weight μ.

Lemma 6.1. — *Let σ be an irreducible representation of $O(m)$.*
(1) *If $\sigma\big|_{SO(m)}$ is irreducible, then σ and $\sigma \otimes \det$ are mutually inequivalent.*
(2) *If $\sigma\big|_{SO(m)}$ is reducible, then σ and $\sigma \otimes \det$ are equivalent. In this case, m is necessarily even. Moreover, there exist positive integers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m/2} > 0$ such that*

$$\sigma\big|_{SO(m)} \simeq \sigma(\mu^+) \oplus \sigma(\mu^-),$$

where $\mu^+ = (\mu_1, \mu_2, \ldots, \mu_{m/2})$ and $\mu^- = (\mu_1, \mu_2, \ldots, -\mu_{m/2})$.

In case (1) in the above lemma, it is subtle to tell the difference between σ and $\sigma \otimes \det$. However, since the difference causes strong influence on our result, we discuss this point.

Take a Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{o}(m)$ as

$$\mathfrak{h}_0 = \{H = \operatorname{diag}(a(\theta_1), a(\theta_2), \ldots, a(\theta_{[m/2]}), 0) \mid \theta_i \in \mathbb{R}\}, \qquad a(\theta) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

where the last 0 in the expression of H appears if and only if m is odd. We define $\varepsilon_j \in \mathfrak{h}^*$ as $\varepsilon_j(H) = \sqrt{-1}\,\theta_j$ in the above expression. Then, positive roots are given by

$$\Delta^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq m/2\} & \text{if } m \text{ is even}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq [m/2]\} \sqcup \{\varepsilon_j \mid 1 \leq j \leq [m/2]\} & \text{if } m \text{ is odd}. \end{cases}$$

Assume that $\sigma\big|_{SO(m)}$ be irreducible. Write $\sigma\big|_{SO(m)} = \sigma(\mu)$ for some highest weight $\mu = \sum_{j=1}^{[m/2]} \mu_j \varepsilon_j$. Let $\delta = \operatorname{diag}(1_{m-1}, -1) \in O(m) \backslash SO(m)$. Then, $\sigma\big|_{SO(m)}$ is irreducible if and only if the twisted representation $\sigma(\mu)^\delta$ is equivalent to $\sigma(\mu)$. Consequently, the highest weight space of $\sigma(\mu)$ is preserved by the action of δ. In particular, if m is even, we get $\mu_{m/2} = 0$.

Since $\delta^2 = 1_m$, its action on the highest weight space is the multiplication by ± 1. If it is 1, we will write $\sigma = \sigma(\mu)$ by abuse of notation; if it is -1, then we denote $\sigma = \sigma(\mu) \otimes \det$. Let $k = \ell(\mu)$ so that $\mu_k > \mu_{k+1} = 0$. We put $\mu^+ = \mu$ if $\sigma = \sigma(\mu)$. If $\sigma = \sigma(\mu) \otimes \det$, we add 1 to μ $(m - 2k)$-times after μ_k, i.e.,

$$\mu^+ = (\mu_1, \mu_2, \ldots, \mu_k, 1, \ldots, 1) = (\mu, 1^{m-2k}).$$

The following theorem is due to Kashiwara-Vergne [**28**] (see also [**23**, § 3.6]).

Theorem 6.2. — *Assume that $m \leq n = \mathbb{R}\text{-}\operatorname{rank} Sp(2n, \mathbb{R})$. Then $L(\sigma)$ is not zero for any $\sigma \in \operatorname{Irr}(O(m))$ and it gives an irreducible unitary lowest weight module of $Sp(2n, \mathbb{R})^\sim$. Let μ^+ be as above, and extend μ^+ to the weight of $Sp(2n, \mathbb{R})$ by adding zero. Then the lowest weight of $L(\sigma)$ is given by*

$$w_K\left(\mu^+ + \frac{m}{2}\mathbb{I}\right),$$

where w_K is the longest element of the Weyl group of $K = U(n)$ and $\mathbb{I} = (1, 1, \ldots, 1)$. Consequently, the lowest K-type of $L(\sigma)$ is $\tau(\mu^+) \otimes \det^{m/2}$.

From this theorem, we get the Poincaré series of $L(\sigma)$ as

$$P(L(\sigma); t^2) = t^{-|\mu^+|} \sum_{\lambda \in \mathcal{P}_m} m(\lambda, \sigma) \dim \tau_\lambda^{(n)} \, t^{|\lambda|}. \tag{6.43}$$

Consider the special case where $\sigma \in \operatorname{Irr}(O(m))$ is the trivial representation $\mathbf{1}_{O(m)}$.

Corollary 6.3. — *We have the K-type decomposition of $L(\mathbf{1}_{O(m)})$ as*

$$L(\mathbf{1}_{O(m)})\big|_{U(n)^\sim} \simeq \sum_{\lambda \in \mathcal{P}_m}^{\oplus} \tau_{2\lambda}^{(n)} \otimes \det^{m/2},$$

where $\mathcal{P}_m$ is the set of all partitions such that $\ell(\lambda) \leq m$. The Poincaré series of $L(\mathbf{1}_{O(m)})$ is given by

$$P(L(\mathbf{1}_{O(m)}); t) = \sum_{\lambda \in \mathcal{P}_m} \dim \tau_{2\lambda}^{(n)} \, t^{|\lambda|}. \tag{6.44}$$

Proof. — It is well-known that

$$m(\lambda, \mathbf{1}_{O(m)}) = \begin{cases} 1 & \text{if } \lambda \text{ is an even partition,} \\ 0 & \text{otherwise.} \end{cases}$$

Apply this formula to (6.42) and (6.43). □

6.2. Case (U, U). — We consider the pair $(G_1, G_2) = (U(p,q), U(m))$. We put $G = G_1 = U(p,q)$ in this subsection. Assume that $m \leq \min(p,q) = \mathbb{R}\text{-rank}\, U(p,q)$. This means that the pair $(U(p,q), U(m))$ is in the stable range.

A maximal compact subgroup of G is isomorphic to $U(p) \times U(q)$, and we realized it as

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in U(p), B \in U(q) \right\} \subset U(p,q). \tag{6.45}$$

Put $n = p + q$. In this case, $K \times G_2$ acts on $M_{n,m}$ in somewhat distorted manner. Let us identify $M_{n,m} = M_{p,m} \oplus M_{q,m}$. Then the action of $\operatorname{diag}(a,b) \times g \in (U(p) \times U(q)) \times U(m)$ is given by

$$M_{p,m} \oplus M_{q,m} \ni X \oplus Y \longmapsto aX\,{}^t g \oplus \overline{b} Y\,{}^t \overline{g}. \tag{6.46}$$

This action gives the projection $K \times G_2 \to K \cdot G_2 \subset \mathcal{K} = U(nm)$, where $\mathcal{K}$ is the maximal compact subgroup of $\mathcal{G} = Sp(2nm, \mathbb{R})$ (cf. (3.19) and (3.20)). The kernel of the projection $K \times G_2 \to K \cdot G_2$ is given by $\{((\alpha 1_p, \alpha 1_q), \alpha^{-1} 1_m) \mid \alpha \in \mathbb{C}^\times, |\alpha| = 1\}$.

Let us consider the Weil representation Ω of $\widetilde{\mathcal{G}} = Mp(2nm, \mathbb{R})$ on $\mathbb{C}[M_{n,m}^*]$. Then the representation of $\Omega\big|_{\widetilde{\mathcal{K}}}$ is isomorphic to $\mathbb{C}[M_{n,m}^*] \otimes \det^{1/2}$. If we consider the underlying space $\mathbb{C}[M_{n,m}^*]$ as $\mathbb{C}[M_{n,m}^*] = \mathbb{C}[M_{p,m}^*] \otimes \mathbb{C}[M_{q,m}]$, the above embedding of $K \times G_2$ into $\mathcal{K}$ tells us that the representation of $U(p) \times U(q)$ is isomorphic to

$$\left(\mathbb{C}[M_{p,m}^*] \otimes \det^{m/2}\right) \boxtimes \left(\mathbb{C}[M_{q,m}] \otimes \det^{-m/2}\right); \tag{6.47}$$

while that of $U(m)$ is isomorphic to

$$\left(\mathbb{C}[M_{p,m}^*]\otimes \det{}^{p/2}\right)\otimes\left(\mathbb{C}[M_{q,m}]\otimes \det{}^{-q/2}\right)\simeq\left(\mathbb{C}[M_{p,m}^*]\otimes\mathbb{C}[M_{q,m}]\right)\otimes \det{}^{(p-q)/2}. \tag{6.48}$$

Therefore the one-dimensional character $\chi\in\mathrm{Irr}(\widetilde{G_2})$ is equal to $\det{}^{(p-q)/2}$, and the one-dimensional character χ_1 of $\widetilde{K}$ is $\det{}^{m/2}\boxtimes\det{}^{-m/2}$.

Let us first consider the untwisted representation $\mathbb{C}[M_{n,m}^*]$ of $\mathcal{K}$. To decompose the restriction to $K\times G_2$, we make use of $U(p)\times U(m)$ (or $U(q)\times U(m)$) duality. We have the decomposition as $(U(p)\times U(q))\times U(m)$-module

$$\begin{aligned}\mathbb{C}[M_{n,m}^*] &= \mathbb{C}[M_{p,m}^*]\otimes\mathbb{C}[M_{q,m}]\\ &= \left(\sum_{\lambda\in\mathcal{P}_m}^{\oplus}\tau_\lambda^{(p)}\boxtimes\tau_\lambda^{(m)}\right)\otimes\left(\sum_{\mu\in\mathcal{P}_m}^{\oplus}(\tau_\mu^{(q)})^*\boxtimes(\tau_\mu^{(m)})^*\right)\\ &= \sum_{\lambda,\mu\in\mathcal{P}_m}^{\oplus}\left(\tau_\lambda^{(p)}\boxtimes(\tau_\mu^{(q)})^*\right)\boxtimes\left(\tau_\lambda^{(m)}\otimes(\tau_\mu^{(m)})^*\right).\end{aligned}$$

Therefore, if we define the branching coefficient $m(\lambda,\mu;\nu)$ by

$$\tau_\lambda^{(m)}\otimes(\tau_\mu^{(m)})^* = \sum_{\nu}^{\oplus} m(\lambda,\mu;\nu)\,\tau_\nu^{(m)}, \tag{6.49}$$

we get

$$\mathbb{C}[M_{n,m}^*]\big|_{K\times U(m)} = \sum_{\nu}^{\oplus}\left\{\sum_{\lambda,\mu\in\mathcal{P}_m}^{\oplus} m(\lambda,\mu;\nu)\,\tau_\lambda^{(p)}\boxtimes(\tau_\mu^{(q)})^*\right\}\boxtimes\tau_\nu^{(m)}.$$

To get the representation $\Omega\big|_{\widetilde{K_1}\times\widetilde{G_2}}$, we should twist the above decomposition by $(\det^{m/2}\boxtimes\det^{-m/2})\boxtimes\det^{(p-q)/2}$. After this twisting, for $\sigma=\tau_\nu^{(m)}\in\mathrm{Irr}(U(m))$, we get the K-type decomposition of $L(\sigma)$:

$$L(\tau_\nu^{(m)})\big|_{\widetilde{K}}\simeq\sum_{\lambda,\mu\in\mathcal{P}_m}^{\oplus} m(\lambda,\mu;\nu)\,(\tau_\lambda^{(p)}\otimes\det{}^{m/2})\boxtimes(\tau_\mu^{(q)}\otimes\det{}^{m/2})^*. \tag{6.50}$$

To determine the lowest weight of $L(\tau_\nu^{(m)})$, we prove a lemma.

Lemma 6.4. — *Take an arbitrary dominant integral weight ν of $U(m)$, and write it as*

$$\nu=(a_1,a_2,\dots,a_s,0,\dots,0,-b_t,\dots,-b_2,-b_1),$$

where

$$\begin{aligned}&a_1\ge a_2\ge\cdots\ge a_s>0,\quad b_1\ge b_2\ge\cdots\ge b_t>0,\\ &a_i,b_j\in\mathbb{Z};\quad s,t\ge 0 \text{ and } s+t\le m.\end{aligned}$$

Consider a set of pairs of partitions $\{(\lambda,\mu) \in \mathcal{P}_m \times \mathcal{P}_m \mid m(\lambda,\mu;\nu) \neq 0\}$. Then partitions

$$\begin{cases} \lambda = \alpha := (a_1, a_2, \dots, a_s, 0, \dots, 0) & \text{and} \\ \mu = \beta := (b_1, b_2, \dots, b_t, 0, \dots, 0) \end{cases}$$

minimize the degree $|\lambda| + |\mu|$ among such pairs. Moreover, (α,β) is a unique pair which attains the minimal degree. In this case, $m(\alpha,\beta;\nu) = 1$ holds.

Proof. — Take a sufficiently large $l \geq 0$ such that $\nu' = \nu + (l, \dots, l)$ becomes a partition. We have

$$\begin{aligned} m(\lambda,\mu;\nu) &= \dim \left((\tau_\lambda \otimes \tau_\mu^*) \otimes \tau_\nu^* \right)^{U(m)} && (6.51) \\ &= \dim \left((\tau_\lambda \otimes \det{}^l) \otimes (\tau_\mu \otimes \tau_{\nu'})^* \right)^{U(m)} && (6.52) \\ &= c_{\mu,\nu'}^{\lambda+(l^m)} \neq 0, && (6.53) \end{aligned}$$

where $c_{\gamma,\delta}^{\eta} = [\tau_\gamma \otimes \tau_\delta : \tau_\eta]$ denotes the Littlewood-Richardson coefficient. Since $c_{\gamma,\delta}^{\eta} \neq 0$ implies $|\eta| = |\gamma| + |\delta|$, we have $|\lambda| + ml = |\mu| + |\nu'|$, or equivalently $|\lambda| = |\mu| + |\nu|$. Therefore, in order to minimize $|\lambda| + |\mu|$, we only have to make $|\lambda|$ minimal. However, if ν' is not contained in $\lambda + (l^m)$, the coefficient $c_{\mu,\nu'}^{\lambda+(l^m)}$ vanishes. Therefore, $\lambda = \alpha$ is the smallest possible partition (e.g., see [**13**, § 5.2, Proposition 3]). If we take $\mu = \beta$, then it is easy to see that $m(\alpha,\beta;\nu) = 1$ (loc. cit.).

If we denote the highest weight of τ_ν^* by

$$\nu^* = (b_1, b_2, \dots, b_t, 0, \dots, 0, -a_s, \dots, -a_2, -a_1),$$

it holds that $m(\lambda,\mu;\nu) = m(\mu,\lambda;\nu^*)$. By the same argument as above, we conclude that $\mu = \beta$ is the only possibility for $m(\alpha,\beta;\nu) \neq 0$. □

Remark 6.5. — If ν is also a partition, the above proof tells us that $m(\lambda,\mu;\nu) = c_{\mu,\nu}^{\lambda}$, where $c_{\mu,\nu}^{\lambda}$ is the Littlewood-Richardson coefficient.

Theorem 6.6. — *Assume that $m \leq \min(p,q) = \mathbb{R}\text{-rank}\, U(p,q)$. Then $L(\sigma)$ is not zero for any $\sigma = \tau_\nu^{(m)} \in \mathrm{Irr}(U(m))$ and it gives an irreducible unitary lowest weight module of $U(p,q)^\sim$. For ν, define α, β as in Lemma 6.4, and put $\beta^* = (0, \dots, 0, -b_t, \dots, -b_2, -b_1)$. Then the lowest weight of $L(\tau_\nu^{(m)})$ is given by*

$$w_K \left(\alpha + \frac{m}{2}\mathbb{I}_p, \beta^* - \frac{m}{2}\mathbb{I}_q \right),$$

where w_K is the longest element of the Weyl group of $K = U(p) \times U(q)$ and $\mathbb{I}_p = (1, \dots, 1) = (1^p)$. Consequently, the lowest K-type of $L(\tau_\nu^{(m)})$ is $(\tau_\alpha^{(p)} \otimes \det^{m/2}) \boxtimes (\tau_\beta^{(q)} \otimes \det^{m/2})^*$.

Proof. — It is known that $L(\sigma)$ is an irreducible unitary lowest weight module of the metaplectic cover $U(p,q)^\sim$. So we simply have to determine its lowest weight. To do that, we only need to know the lowest K-type (or harmonic K-type) which is unique. By Lemma 6.4, we conclude that $\tau_\alpha^{(p)} \boxtimes \tau_\beta^{(q)*}$ gives such a K-type with a twist by $\chi_1 = \det{}^{m/2} \boxtimes \det{}^{-m/2}$. □

By this theorem, we obtain Poincaré series of $L(\tau_\nu^{(m)})$:

$$\begin{aligned} P(L(\tau_\nu^{(m)});t) &= t^{-|\beta|} \sum_{\lambda,\mu\in\mathcal{P}_m} m(\lambda,\mu;\nu) \dim\tau_\lambda^{(p)} \dim\tau_\mu^{(q)}\, t^{|\mu|} \\ &= t^{-|\alpha|} \sum_{\lambda,\mu\in\mathcal{P}_m} m(\lambda,\mu;\nu) \dim\tau_\lambda^{(p)} \dim\tau_\mu^{(q)}\, t^{|\lambda|}. \end{aligned}$$

This formula follows from (5.37) after a reflection on degrees. Note that the summation is taken over (λ,μ) satisfying $|\lambda|-|\mu|=|\nu|$ (see the proof of Lemma 6.4). Hence the total degree of $\tau_\lambda^{(p)} \boxtimes (\tau_\mu^{(q)})^*$ is given by $|\lambda|+|\mu| = 2|\mu|+|\nu| = 2|\lambda|-|\nu|$, while $|\nu| = |\alpha|-|\beta|$ and $k_\sigma = |\alpha|+|\beta|$ for $\sigma = \tau_\nu^{(m)}$.

Consider the special case where $\tau_\nu^{(m)}$ is trivial, i.e., $\nu=0$. Then it is easy to see that

$$m(\lambda,\mu;0) = \begin{cases} 1 & \text{if } \lambda=\mu, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get

Corollary 6.7. — *We have the K-type decomposition of $L(\mathbf{1}_{U(m)})$ as*

$$L(\mathbf{1}_{U(m)})\big|_{\widetilde{K}} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} (\tau_\lambda^{(p)} \otimes \det{}^{m/2}) \boxtimes (\tau_\lambda^{(q)} \otimes \det{}^{m/2})^*. \tag{6.54}$$

Its Poincaré series becomes

$$P(L(\mathbf{1}_{U(m)});t) = \sum_{\lambda\in\mathcal{P}_m} \dim\tau_\lambda^{(p)} \dim\tau_\lambda^{(q)}\, t^{|\lambda|}. \tag{6.55}$$

6.3. Case (O^*,Sp). — We consider the pair $(G_1,G_2) = (O^*(2p), Sp(2m))$ in $\mathcal{G} = Sp(2nm,\mathbb{R})$ $(n=2p)$, which is in the stable range, i.e., we assume that $m \le [p/2] = \mathbb{R}\text{-rank}\, O^*(2p)$.

In this case, a maximal compact subgroup K of $O^*(2p)$ is isomorphic to $U(p)$. We realize the isomorphism as

$$O^*(2p) \supset K = \left\{ \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix} \mid X \in U(p) \right\} \leftrightarrow X \in U(p). \tag{6.56}$$

Then $K\times G_2$ is imbedded into $\mathcal{K} = U(2pm)$ canonically. To be more precise, this embedding of $K=U(p)$ and $G_2 = Sp(2m) = Sp(2m,\mathbb{C})\cap U(2m)$ is given by the action on $M_{p,2m}$ as

$$M_{p,2m} \ni X \longmapsto gX\,{}^t h \qquad ((g,h)\in K\times G_2).$$

The action induces a projection $K\times G_2 \to K\cdot G_2 \subset \mathcal{K}$ with kernel $\{(\pm 1_p, \pm 1_{2m}) \in U(p)\times Sp(2m)\}$.

Let us consider the Weil representation Ω. As the representation space of Ω, we take the polynomial ring $\mathbb{C}[M_{p,2m}^*]$ as before. Then we know that $\Omega\big|_{\widetilde{K}}$ is isomorphic to

$$\mathbb{C}[M_{p,2m}^*] \otimes \det{}^m,$$

where $\mathbb{C}[M^*_{p,2m}]$ is considered as the symmetric tensor product of the representation $M_{p,2m}$ above. On the other hand, we have $\widetilde{G_2} \simeq Sp(2m) \times \mathbb{Z}_2$, and the one-dimensional character χ arises as the non-trivial character of $\mathbb{Z}_2$ as in (6.40). Therefore, we have $\Omega\big|_{\widetilde{G_2}} \simeq \mathbb{C}[M^*_{p,2m}] \otimes \chi$.

First, let us treat the untwisted symmetric tensor. So we decompose $\mathbb{C}[M^*_{p,2m}]$ by using $U(p) \times U(2m)$ duality :

$$\mathbb{C}[M^*_{p,2m}]\big|_{U(p)\times U(2m)} \simeq \sum_{\lambda\in\mathcal{P}_{2m}}^{\oplus} \tau^{(p)}_\lambda \boxtimes \tau^{(2m)}_\lambda.$$

Take a highest weight λ for $U(2m)$ and μ for $Sp(2m)$. Let us define the branching coefficient $m(\lambda, \mu)$ by

$$\tau^{(2m)}_\lambda\big|_{Sp(2m)} \simeq \sum_{\mu}^{\oplus} m(\lambda,\mu)\sigma_\mu,$$

where $\sigma_\mu \in \mathrm{Irr}(Sp(2m))$ is the irreducible representation of $Sp(2m)$ with highest weight μ. We also write $m(\lambda, \sigma_\mu)$ instead of $m(\lambda, \mu)$. With this notation, we can write down the decomposition :

$$\mathbb{C}[M^*_{p,2m}]\big|_{U(p)\times Sp(2m)} = \sum_{\sigma_\mu\in\mathrm{Irr}(Sp(2m))}^{\oplus} \left\{ \sum_{\lambda\in\mathcal{P}_{2m}}^{\oplus} m(\lambda,\mu)\, \tau^{(p)}_\lambda \right\} \boxtimes \sigma_\mu.$$

To get the restricted representation $\Omega\big|_{\widetilde{K}\times\widetilde{G_2}}$, we must twist the above representation by $\det^m \boxtimes \chi$. Therefore $L(\sigma)\big|_{\widetilde{K}}$ decomposes as

$$L(\sigma_\mu)\big|_{U(p)^\sim} \simeq \sum_{\lambda\in\mathcal{P}_{2m}}^{\oplus} m(\lambda,\mu)\, \tau^{(p)}_\lambda \otimes \det{}^m. \tag{6.57}$$

This formula describes the multiplicities of K-types of $L(\sigma_\mu)$ in the case of Case (O^*, Sp).

Theorem 6.8. — *Assume that $m \leq [p/2] = \mathbb{R}\text{-rank}\, O^*(2p)$. Then $L(\sigma)$ is not zero for any $\sigma = \sigma_\mu \in \mathrm{Irr}(Sp(2m))$ and it gives an irreducible unitary lowest weight module of $O^*(2p)^\sim$. Extend μ to the weight of $O^*(2p)$ by adding zero. Then the lowest weight of $L(\sigma_\mu)$ is given by*

$$w_K\,(\mu + m\mathbb{I}_p)\,,$$

where w_K is the longest element of the Weyl group of $K = U(p)$ and $\mathbb{I}_p = (1, \ldots, 1) = (1^p)$. Consequently, the lowest K-type of $L(\sigma_\mu)$ is $\tau^{(p)}_\mu \otimes \det{}^m$.

Proof. — See [**23**, § 3.8.5]. □

From the above theorem, we obtain the Poincaré series of $L(\sigma_\mu)$:

$$P(L(\sigma_\mu); t^2) = t^{-|\mu|} \sum_{\lambda\in\mathcal{P}_{2m}} m(\lambda,\mu)\, \dim \tau^{(p)}_\lambda\, t^{|\lambda|}.$$

Consider the special case where $\sigma_\mu = \mathbf{1}_{Sp(2m)}$, i.e., $\mu = 0$. It is well-known that

$$m(\lambda, \mathbf{1}_{Sp(2m)}) = \begin{cases} 1 & \text{if } \lambda_{2i-1} = \lambda_{2i} \text{ for } 1 \leq i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

So we get

Corollary 6.9. — *We have the K-type decomposition of $L(\mathbf{1}_{Sp(2m)})$ as*

$$L(\mathbf{1}_{Sp(2m)})\big|_{\widetilde{K}} \simeq \sum_{\lambda \in \mathcal{P}_m}^{\oplus} \tau_{\lambda^\#}^{(p)} \otimes \det{}^m, \tag{6.58}$$

where $\lambda^\# = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$ is a transposed even partition which is obtained by doubling each row of λ. Its Poincaré series is given by

$$P(L(\mathbf{1}_{Sp(2m)}); t) = \sum_{\lambda \in \mathcal{P}_m} \dim \tau_{\lambda^\#}^{(p)}\, t^{|\lambda|}. \tag{6.59}$$

7. Degree of nilpotent orbits

7.1. Automorphism groups of Hermitian symmetric spaces. — Let G be one of real reductive Lie groups $Sp(2n, \mathbb{R})$, $U(p, q)$, or $O^*(2p)$. These groups appear as the group G_1 in Table 2. The division algebra D is specified there. Let K be a maximal compact subgroup of G specified in § 6. In all cases, the corresponding Riemannian symmetric spaces G/K have G-invariant complex structure. In other words, the spaces G/K are Hermitian symmetric spaces. For $G = Sp(2n, \mathbb{R})$, $U(p, p)$, or $O^*(4k)$, the corresponding Hermitian symmetric space G/K is of *tube type*. For $G = U(p, q)$ with $p \neq q$, or $O^*(4k + 2)$, G/K is not of tube type. For definitions and properties of symmetric spaces, see [**20**].

We fix a complexification $G_{\mathbb{C}}$ of the real Lie group G. Let $K_{\mathbb{C}}$ be the minimal complex Lie subgroup of $G_{\mathbb{C}}$ containing K. We list up here $(G_{\mathbb{C}}, K_{\mathbb{C}})$ for the convenience of readers.

Table 3. Complexifications of (G, K).

G	$G_{\mathbb{C}}$	$K_{\mathbb{C}}$
$Sp(2n, \mathbb{R})$	$Sp(2n, \mathbb{C})$	$GL(n, \mathbb{C})$
$U(p, q)$	$GL(p + q, \mathbb{C})$	$GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$
$O^*(2p)$	$O(2p, \mathbb{C})$	$GL(p, \mathbb{C})$

For real Lie groups such as G and K, we denote the corresponding Lie algebra by $\mathfrak{g}_0$, $\mathfrak{k}_0$, respectively. Its complexification is denoted by $\mathfrak{g}$ and $\mathfrak{k}$. The corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is stable under the restriction of the adjoint action to $K_{\mathbb{C}}$. Moreover, in our cases, the subspace $\mathfrak{p}$ breaks up into the sum of two non-isomorphic irreducible representations of $K_{\mathbb{C}}$, say

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

The representation $\mathfrak{p}^-$ is the contragredient representation of $\mathfrak{p}^+$.

Let us describe the pair $(\mathfrak{p}^+, \mathfrak{p}^-)$ and the action of $K_{\mathbb{C}}$ on them for each case. Although the action itself is fairly well-known, we need more explicit features in the following.

For $G = Sp(2n, \mathbb{R})$, we realize it as in (4.23) and a maximal compact subgroup $K \simeq U(n)$ is also specified there (4.24). The complexification $G_{\mathbb{C}}$ is identified naturally with $Sp(2n, \mathbb{C})$ with respect to the same symplectic form as G (see (3.13) for the symplectic form). Then, the decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is given by

$$\mathfrak{p}^{\pm} = \left\{ \begin{pmatrix} \pm\sqrt{-1}\, A & A \\ A & \mp\sqrt{-1}\, A \end{pmatrix} \mid A \in \mathrm{Sym}\,(n, \mathbb{C}) \right\}.$$

Therefore, we can identify the both spaces with the space of symmetric matrices of size n. To see the action of $K_{\mathbb{C}} \simeq GL(n, \mathbb{C})$, it is more convenient to use the different realization of $Sp(2n, \mathbb{R})$. Let

$$\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_n & -\sqrt{-1}\, 1_n \\ -\sqrt{-1}\, 1_n & 1_n \end{pmatrix},$$

which is called the Cayley transform. The conjugation of $Sp(2n,\mathbb{R})$ by γ produces a different (but isomorphic) real form of $Sp(2n,\mathbb{C})$, and we denote it by $G^\gamma = Sp(2n,\mathbb{R})^\gamma$. In G^γ, the conjugated maximal compact subgroup K^γ has a simple diagonal form:

$$K^\gamma = \left\{ \begin{pmatrix} k & 0 \\ 0 & {}^t k^{-1} \end{pmatrix} \mid k \in U(n) \right\}.$$

The complexification $K^\gamma_\mathbb{C}$ is also expressed similarly as above, but k belonging to $GL(n,\mathbb{C})$. Then $\mathfrak{p}^\gamma$ is represented by off diagonal matrices

$$\mathfrak{p}^\gamma = \left\{ \left(\begin{array}{cc} 0 & B \\ C & 0 \end{array} \right) \mid B, C \in \mathrm{Sym}\,(n,\mathbb{C}) \right\},$$

and

$$\mathfrak{p}^{\gamma+} = \left\{ \left(\begin{array}{cc} 0 & B \\ 0 & 0 \end{array} \right) \mid B \in \mathrm{Sym}\,(n,\mathbb{C}) \right\}, \qquad \mathfrak{p}^{\gamma-} = \left\{ \left(\begin{array}{cc} 0 & 0 \\ C & 0 \end{array} \right) \mid C \in \mathrm{Sym}\,(n,\mathbb{C}) \right\}.$$

We denote the element $\left(\begin{array}{cc} 0 & B \\ C & 0 \end{array} \right)$ of $\mathfrak{p}^\gamma$ by (B,C). Then the adjoint action of an element $k \in K^\gamma_\mathbb{C}$ on $\mathfrak{p}^\gamma$ is given by

$$k(B,C) = (kB\,{}^t k,\ {}^t k^{-1} C k^{-1}).$$

We sometimes identify the $K_\mathbb{C}$-module $\mathfrak{p}^+$ with $\mathrm{Sym}\,(n,\mathbb{C})$.

For $G = U(p,q)$, we realized it as the full isometry group of the indefinite Hermitian form (3.17) (cf. (3.18)), and a maximal compact subgroup $K \simeq U(p) \times U(q)$ is given in (6.45). The complexification $G_\mathbb{C}$ is naturally identified with $GL(p+q,\mathbb{C})$, and $K_\mathbb{C}$ is given by

$$K_\mathbb{C} = \left\{ k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \mid k_1 \in GL(p,\mathbb{C}),\ k_2 \in GL(q,\mathbb{C}) \right\}.$$

The other member of the Cartan decomposition is expressed by off diagonal matrices

$$\mathfrak{p} = \left\{ \left(\begin{array}{cc} 0 & B \\ C & 0 \end{array} \right) \mid B \in M(p,q,\mathbb{C}), C \in M(q,p,\mathbb{C}) \right\},$$

and such an element is denoted by (B,C). Irreducible subspaces $\mathfrak{p}^\pm$ are given as

$$\mathfrak{p}^+ = \left\{ \left(\begin{array}{cc} 0 & B \\ 0 & 0 \end{array} \right) \mid B \in M(p,q,\mathbb{C}) \right\}, \qquad \mathfrak{p}^- = \left\{ \left(\begin{array}{cc} 0 & 0 \\ C & 0 \end{array} \right) \mid C \in M(q,p,\mathbb{C}) \right\}.$$

The adjoint action of an element $k = (k_1, k_2) \in K_\mathbb{C}$ on $\mathfrak{p}$ is given by

$$(k_1,k_2)(B,C) = (k_1 B k_2^{-1}, k_2 C k_1^{-1}).$$

Therefore, the representation $\mathfrak{p}^+$ of $K_\mathbb{C}$ is identified with $M(p,q,\mathbb{C})$.

For $G = O^*(2p)$, we gave a realization in (3.21). A maximal compact subgroup $K \simeq U(p)$ is chosen again as diagonal matrices (6.56). The complexified Lie group $G_\mathbb{C}$ is identified with

$$O(2p,\mathbb{C}) = \left\{ Z \in GL(2p,\mathbb{C}) \mid {}^t Z S_p Z = S_p \right\}, \qquad S_p = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}, \tag{7.60}$$

and

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} k & 0 \\ 0 & {}^t k^{-1} \end{pmatrix} \mid k \in GL(p, \mathbb{C}) \right\}.$$

We identify $K_{\mathbb{C}}$ and $GL(p, \mathbb{C})$ in the following, so $k \in K_{\mathbb{C}}$ denotes a matrix in $GL(p, \mathbb{C})$. Now $\mathfrak{p}$ becomes

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in \mathrm{Alt}\,(p, \mathbb{C}) \right\}.$$

As above, we denote the element $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ by (B, C). The $K_{\mathbb{C}}$ stable decomposition of $\mathfrak{p}$ is given by

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in \mathrm{Alt}\,(p, \mathbb{C}) \right\}, \qquad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in \mathrm{Alt}\,(p, \mathbb{C}) \right\}.$$

The adjoint action of an element $k \in K_{\mathbb{C}}$ on $\mathfrak{p}$ is

$$k(B, C) = (kB{}^t k, {}^t k^{-1} C k^{-1}).$$

We identify the $K_{\mathbb{C}}$-module $\mathfrak{p}^+$ with $\mathrm{Alt}\,(p, \mathbb{C})$.

7.2. Kostant-Rallis decomposition. — In this subsection, we summarize the $K_{\mathbb{C}}$-orbit decomposition of $\mathfrak{p}^-$. The orbit decomposition of $\mathfrak{p}^+$ is the same. We denote the real rank of Lie group G by r (cf. § 6). Then there are exactly $(r+1)$ $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$ in each of the above three cases. We give a parametrization of these orbits by $(r+1)$ integers, $0, 1, \ldots, r$,

$$\mathfrak{p}^- = \coprod_{j=0}^{r} \mathcal{O}_j.$$

We know that the dimension of the orbits are distinct (see below). We arrange the numbering of orbits so that an orbit with the larger index has the larger dimension. With this indexing, the set $\mathcal{O}_r$ is an open dense subset of $\mathfrak{p}^-$ in the classical topology (or, also in Zariski topology). On the contrary, the orbit $\mathcal{O}_0 = \{0\}$. We also know that the closure in classical topology (or, also in Zariski topology),

$$\overline{\mathcal{O}_m} = \coprod_{j \leq m} \mathcal{O}_j.$$

In other words, the closure relation of the orbits is linear ordering. Each closure is a Zariski closed subset of the affine space $\mathfrak{p}^-$, then $\overline{\mathcal{O}_m}$ is an affine algebraic variety. We denote the defining ideal of these subset $\overline{\mathcal{O}_m}$ by

$$I_m := \{p \in \mathbb{C}[\mathfrak{p}^-] \mid p|_{\overline{\mathcal{O}_m}} = 0\}.$$

This is an ideal of the polynomial ring $\mathbb{C}[\mathfrak{p}^-]$ on $\mathfrak{p}^-$. Then, by definition, the coordinate ring $\mathbb{C}[\overline{\mathcal{O}_m}]$ is isomorphic to the residual ring $\mathbb{C}[\mathfrak{p}^-]/I_m$.

Note that we can identify $\mathfrak{p}^+$ with the dual vector space of $\mathfrak{p}^-$ via Killing form. Therefore, by the natural identification, $\mathbb{C}[\mathfrak{p}^-] = S(\mathfrak{p}^+)$, where $S(\mathfrak{p}^+)$ denotes the symmetric algebra. Since $\mathfrak{p}^+$ is an abelian subspace of $\mathfrak{g}$, we also identify $S(\mathfrak{p}^+)$ with the enveloping algebra $U(\mathfrak{p}^+)$. We use these identification freely in the following. In particular, as a $K_{\mathbb{C}}$-module, $\mathbb{C}[\overline{\mathcal{O}_m}]$ is isomorphic to a quotient module of $S(\mathfrak{p}^+)$.

Let $L = L(\sigma)$ be an irreducible unitary lowest weight module of G treated in § 5. We construct a good filtration of L by taking the lowest K-type as the generating subspace of L (cf. § 1.2). Let $M = \operatorname{gr} L$ be the associated graded $S(\mathfrak{g})$-module. Since the generating subspace is preserved by $\mathfrak{k}$ and $\mathfrak{p}^-$, the $S(\mathfrak{g})$-module M is annihilated by $\mathfrak{k}$ and $\mathfrak{p}^-$. Therefore, its associated variety $\mathcal{AV}(L)$ is contained in $\mathfrak{p}^-$, by the identification above, and is a $K_{\mathbb{C}}$ stable closed subset. Since the $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$ has linear ordering with respect to the closure relation, we can conclude that $\mathcal{AV}(L) = \overline{\mathcal{O}_m}$ for some $0 \leq m \leq r$.

In the following subsections, we see that there is a strong relationship between $\mathbb{C}[\overline{\mathcal{O}_m}]$ and the K-type decomposition of $L(\mathbf{1}_{G_2})$. In fact, they are the same as $K_{\mathbb{C}}$-modules up to some character. This relationship is an example of general phenomenon and is well-known among experts. It is a part of Vogan's philosophy of orbit method [**52**].

Our main aim of the following subsections is calculation of the Bernstein degree of $L(\mathbf{1}_{G_2})$. Our approach is purely representation theoretic. It turns out that $\operatorname{Deg} L(\mathbf{1}_{G_2})$ coincides with the classical degree of the corresponding orbit $\overline{\mathcal{O}_m} = \mathcal{AV}(L(\mathbf{1}_{G_2}))$, which coincides with determinantal variety of various type (see, e.g, [**12**] or [**16**, Lecture 9]). Hence our calculation here will give a new proof of the formula of $\deg \overline{\mathcal{O}_m}$ called *Giambelli-Thom-Porteous formula* ([**15**], [**17**]; also see [**12**, Chapter 14]).

7.3. The case $G = Sp(2n, \mathbb{R})$. — Consider $G = Sp(2n, \mathbb{R})$. In this case, as is given above, $K = U(n)$, $K_{\mathbb{C}} = GL(n, \mathbb{C})$, $\mathfrak{p}^- = \operatorname{Sym}(n, \mathbb{C})$. The action of $k \in K_{\mathbb{C}}$ on $A \in \mathfrak{p}^-$ is given by

$$k \cdot A = {}^t k^{-1} A k^{-1} \qquad (k \in GL(n, \mathbb{C}), A \in \operatorname{Sym}(n, \mathbb{C})). \tag{7.61}$$

We define a locally closed subset of $\operatorname{Sym}(n, \mathbb{C})$ by

$$\mathcal{O}_m = \{A \in \operatorname{Sym}(n, \mathbb{C}) \mid \operatorname{rank}(A) = m\}, \qquad (m = 0, 1, \ldots, n).$$

By the definition of the action of $K_{\mathbb{C}}$, it is easy to see that $\mathcal{O}_m$ is stable under the action of $K_{\mathbb{C}}$. Moreover, they classify all the $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$. The matrix $\sum_{j=1}^m E_{jj}$ belongs to the orbit $\mathcal{O}_m$. Here, E_{ij} is the matrix unit, that is, (i, j)-entry of the matrix E_{ij} is one and all other entries are zero. The dimension of the orbit $\mathcal{O}_m$ is given by

$$\dim \mathcal{O}_m = rm - (m-1)m/2.$$

For subsets $I = \{i_1, i_2, \ldots, i_{m+1}\}$ and $J = \{j_1, j_2, \ldots, j_{m+1}\}$ of $\{1, 2, \ldots, n\}$ with the same cardinality $(m+1)$, we define the minor

$$D_{IJ}(A) = \det(a_{i_p j_q})_{1 \leq p, q \leq m+1},$$

where $A = (a_{ij})_{1 \leq i, j \leq n} \in \operatorname{Sym}(n, \mathbb{C})$. Then the defining ideal I_m of $\overline{\mathcal{O}_m}$ is generated by these minors

$$\{D_{IJ} \mid I, J \subset \{1, 2, \ldots, n\}, |I| = |J| = m+1\}.$$

Recall the dual pair $(Sp(2n,\mathbb{R}),O(m))$ in § 3. We define an *unfolding* of the orbit $\mathcal{O}_m$ by an extra action of $O(m)$, or more precisely, its complexification $O(m,\mathbb{C})$. Let us consider the space of $m\times n$ matrices $M_{m,n}=M(m,n,\mathbb{C})$ and define an action of $K_{\mathbb{C}}\times O(m,\mathbb{C})=GL(n,\mathbb{C})\times O(m,\mathbb{C})\ni(k,h)$ on $M_{m,n}$ by

$$(k,h)\cdot X=hXk^{-1}\qquad(X\in M_{m,n}).$$

For $X\in M_{m,n}$, we define

$$\psi(X)={}^tXX\in\mathrm{Sym}\,(n,\mathbb{C}).$$

This is a polynomial map of degree two. With the trivial action of $O(m,\mathbb{C})$ on $\overline{\mathcal{O}_m}$, the map

$$\psi:M_{m,n}\to\overline{\mathcal{O}_m}$$

is $K_{\mathbb{C}}\times O(m,\mathbb{C})$-equivariant, that is, $\psi(hXk^{-1})={}^tk^{-1}\psi(X)k^{-1}$ for all $k\in K_{\mathbb{C}}$, $h\in O(m,\mathbb{C})$. We see that the image of ψ coincides with $\overline{\mathcal{O}_m}$.

Lemma 7.1. — *The map ψ above induces the $\mathbb{C}$-algebra isomorphism*

$$\psi^*:\mathbb{C}[\overline{\mathcal{O}_m}]\ni f\mapsto f\circ\psi\in\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}=S(M_{n,m})^{O(m,\mathbb{C})},$$

which means that $\overline{\mathcal{O}_m}$ is the geometric quotient $M_{n,m}//O(m,\mathbb{C})$. In particular, $\overline{\mathcal{O}_m}$ is a normal variety. Here we consider $M_{m,n}=M_{n,m}^*$ as the algebraic dual of $M_{n,m}$.

Proof. — The induced map ψ^* is injective since ψ is surjective. The classical invariant theory, in the modern reformulation [**24**, § 3.4], says that every $O(m,\mathbb{C})$-invariants on $M_{m,n}$ is generated by typical invariants of degree two, which implies the map ψ^* is surjective. □

Now we come back to the dual pair $(Sp(2n,\mathbb{R}),O(m))$ in $\mathcal{G}=Sp(2nm,\mathbb{R})$ and the Weil representation Ω of $\widetilde{\mathcal{G}}$ (cf. § 3). Let $L(\mathbf{1}_{O(m)})$ be an irreducible unitary lowest weight module of $\widetilde{G}$ which corresponds to the trivial representation of $O(m)$. We should clarify the relationship between $\mathcal{O}_m$ and the representation $L(\mathbf{1}_{O(m)})$.

Since the associated variety of $L(\mathbf{1}_{O(m)})$ is contained in $\mathfrak{p}^-$, it is enough to see the annihilator of $\mathrm{gr}\,L(\mathbf{1}_{O(m)})$ in $U(\mathfrak{p}^+)$. Therefore, let us see the action of the non-compact root vector $X_{\varepsilon_a+\varepsilon_b}\in\mathfrak{p}^+$ via $\Omega:\mathfrak{sp}(2nm,\mathbb{R})\to\mathrm{End}_{\mathbb{C}}(\mathbb{C}[M_{n,m}])$,

$$\Omega(X_{\varepsilon_a+\varepsilon_b})=\frac{1}{2}\sum_{j=1}^{m}x_{aj}x_{bj},$$

(see (4.30)). By this formula, we see $\Omega(X_{\varepsilon_a+\varepsilon_b})\in\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$. Moreover, we have

$$2\,\Omega(X_{\varepsilon_a+\varepsilon_b})=\psi_{ab},$$

here $\psi_{ab}\in\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ is the ab-component of ψ. This means that the subspace spanned by typical invariants ψ_{ab} coincides with the image $\Omega(\mathfrak{p}^+)$. Thus, the subalgebra of $\Omega(U(\mathfrak{p}^+))$ generated by $\Omega(\mathfrak{p}^+)$ is isomorphic to $\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$, which is generated by typical invariants as is explained above. Let us define the natural good filtration of $L = L(\mathbf{1}_{O(m)})$ by $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$, where $\mathbf{1}$ is the constant polynomial with value 1. Then we have an isomorphism

$$L(\mathbf{1}_{O(m)}) \cong \operatorname{gr} L(\mathbf{1}_{O(m)}) \cong U(\mathfrak{p}^+)/I_m$$

as $U(\mathfrak{p}^+)$-modules, $K_{\mathbb{C}}$-modules and filtered modules. The filtration induced by the degree of polynomials coincides with the natural filtration up to a shift. This implies

Lemma 7.2. — *There are algebra isomorphisms*

$$\Omega(U(\mathfrak{p}^+)) \simeq \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})} \simeq \mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathfrak{p}^-]/I_m.$$

We have $\operatorname{Ann} L(\mathbf{1}_{O(m)}) = \operatorname{Ann} \operatorname{gr} L(\mathbf{1}_{O(m)}) = I_m$ *in* $U(\mathfrak{p}^+)$.

Proof. — As is explained above, we have the desired isomorphisms. For the annihilator, note that the representation space $\mathbb{C}[M_{n,m}^*]^{O(m,\mathbb{C})}$ of $L(\mathbf{1}_{O(m)})$ has the natural grading as K-module and $\mathfrak{p}^+$ acts on $\mathbb{C}[M_{n,m}^*]^{O(m,\mathbb{C})} = \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ as a homogeneous operator of degree two. This means that the annihilator in $U(\mathfrak{p}^+)$ does not change after taking gradation as a filtered $U(\mathfrak{g})$-module. □

Corollary 7.3. — *The representation $L(\mathbf{1}_{O(m)})$ has the following properties.*

(1) *The associated variety of $L(\mathbf{1}_{O(m)})$ is $\overline{\mathcal{O}_m}$.*
(2) *As a $\widetilde{K}$-module, $L(\mathbf{1}_{O(m)})$ is isomorphic to $\mathbb{C}[\overline{\mathcal{O}_m}] \otimes \det^{m/2}$.*
(3) *The Bernstein degree of $L(\mathbf{1}_{O(m)})$ coincides with* $\deg \overline{\mathcal{O}_m}$.

Proof. — (1) is a direct consequence of the above lemma.

Let us consider (2). By definition, $L(\mathbf{1}_{O(m)})$ is realized on $\mathbb{C}[M_{n,m}^*]^{O(m,\mathbb{C})} = \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ (see § 5). As is explained in § 5, to get $\widetilde{K}$-module structure of $L(\mathbf{1}_{O(m)})$, we must twist $\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ by $\det^{m/2}$. Therefore, untwisting of $L(\mathbf{1}_{O(m)})$ produces $\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ itself, and the module structure factors through to that of K.

Since the unfolding map ψ has degree two, it is easy to see the definition of $\operatorname{Deg} L(\mathbf{1}_{O(m)})$ and $\deg \overline{\mathcal{O}_m}$ coincides, which proves (3). □

Let us calculate $\operatorname{Deg} L(\mathbf{1}_{O(m)}) = \deg \overline{\mathcal{O}_m}$ explicitly. Recall the good filtration $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$. By (6.44) and the Weyl's dimension formula, we know

$$\begin{aligned} \dim L_k &= \sum_{\lambda \in \mathcal{P}_m, |\lambda| \le k} \dim \tau_{2\lambda}^{(n)} \\ &= \sum_{\lambda \in \mathcal{P}_m, |\lambda| \le k} \frac{\prod_{1 \le i < j \le n} (2\lambda_i - 2\lambda_j - i + j)}{\prod_{1 \le i < j \le n} (j - i)} \end{aligned}$$

$$= \frac{2^{m(m-1)/2+m(n-m)} k^{m(m-1)/2+m(n-m)+m}}{\prod_{i=1}^{m}(n-i)!} \times$$
$$\times \int_{\substack{0 \le x_m \le x_{m-1} \le \cdots \le x_1, \\ x_1+\cdots+x_m \le 1}} \prod_{1 \le i < j \le m} (x_i - x_j) \prod_{i=1}^{m} x_i^{n-m} dx_1 \cdots dx_m$$
$$+ \text{(lower order terms of } k)$$
$$= \frac{2^{mn-m(m+1)/2} k^{mn-m(m-1)/2}}{m! \prod_{i=1}^{m}(n-i)!} \times$$
$$\times \int_{\substack{x_i \ge 0 \\ x_1+\cdots+x_m \le 1}} \prod_{1 \le i < j \le m} |x_i - x_j| \prod_{i=1}^{m} x_i^{n-m} dx_1 \cdots dx_m$$
$$+ \text{(lower order terms of } k)$$

for sufficiently large k. Here, in the third equality, we devide the formula by a suitable power of k and interprete the leading term as a Riemann sum for the integral.

Let us generalize the integral above slightly, and denote it as

$$I^{\alpha}(s,m) = \int_{\substack{x_i \ge 0, \\ x_1+\cdots+x_m \le 1}} |\Delta|^{\alpha} \left(\prod_{i=1}^{m} x_i\right)^{s} dx_1 \cdots dx_m, \tag{7.62}$$

where $\Delta = \prod_{1 \le i < j \le m}(x_i - x_j)$ is the difference product. An explicit formula of this integral is given by using Gamma function of Hermitian symmetric cone ([**37**]).

Theorem 7.4. — *Let $I^{\alpha}(s,m)$ be as in* (7.62). *For* $\operatorname{Re} s > -1$ *and* $\alpha = 1, 2, 4$, *we have*

$$I^{\alpha}(s,m) = \frac{\prod_{j=1}^{m} \Gamma(j\alpha/2+1)\Gamma(s+1+(j-1)\alpha/2)}{\Gamma(\alpha/2+1)^m \Gamma(sm+N+1)}, \tag{7.63}$$

where $N = m + \dfrac{\alpha}{2} m(m-1)$.

Summarizing above, we have the following theorem.

Theorem 7.5. — *Assume that* $m \le n = \mathbb{R}\text{-rank}\, Sp(2n,\mathbb{R})$, *and consider the reductive dual pair* $(Sp(2n,\mathbb{R}), O(m))$.

(1) *The unitarizable lowest weight module* $L(\mathbf{1}_{O(m)})$ *of* $Sp(2n,\mathbb{R})^{\sim}$ *has the lowest weight* $\frac{m}{2}(1,1,\ldots,1) = \frac{m}{2}\sum_{i=1}^{n} \varepsilon_i$. *Its associated cycle is multiplicity-free and given by* $\mathcal{AC}(L(\mathbf{1}_{O(m)})) = [\overline{\mathcal{O}_m}]$.

(2) *The Gelfand-Kirillov dimension and the Bernstein degree of* $L(\mathbf{1}_{O(m)})$ *are*

$$\begin{aligned} \operatorname{Dim} L(\mathbf{1}_{O(m)}) &= \dim \overline{\mathcal{O}_m} = m\left(n - \frac{m-1}{2}\right), \\ \operatorname{Deg} L(\mathbf{1}_{O(m)}) &= \deg \overline{\mathcal{O}_m} = \prod_{l=0}^{m-1} \frac{l!}{l!!} \frac{(2n-2m+l)!!}{(n-m+l)!}, \end{aligned}$$

where $l!! = l(l-2)(l-4)\cdots 2$ *for an even integer* l, *and* $l!! = l(l-2)(l-4)\cdots 1$ *for odd* l.

Proof. — From the top degree term of $\dim L_k$ above, we get the Gelfand-Kirillov dimension

$$\operatorname{Dim} L(\mathbf{1}_{O(m)}) = mn - \frac{m(m-1)}{2} =: d,$$

and

$$\begin{aligned}
\operatorname{Deg} L(\mathbf{1}_{O(m)}) &= \frac{2^{d-m} d!}{m! \prod_{i=1}^{m} (n-i)!} I^1(n-m, m) \\
&= \frac{2^d}{\pi^{m/2} m!} \prod_{j=1}^{m} \frac{\Gamma(j/2+1)\Gamma(d/m-(j-1)/2)}{\Gamma(n-j+1)} \\
&= \prod_{l=0}^{m-1} \frac{l!}{l!!} \frac{(2n-2m+l)!!}{(n-m+l)!}.
\end{aligned}$$

□

We close this subsection by giving the relation between the lowest weight module $L(\mathbf{1}_{O(m)})$ and the half-form bundle on the orbit $\mathcal{O}_m$. We choose a representative

$$\lambda = \sum_{j=1}^{m} E_{jj} \in \mathcal{O}_m \subset \operatorname{Sym}(n, \mathbb{C}) \cong \mathfrak{p}^-$$

of the orbit $\mathcal{O}_m$. The group $K_{\mathbb{C}} = GL(n, \mathbb{C})$ acts on $\mathcal{O}_m$ transitively by (7.61). The stabilizer $(K_{\mathbb{C}})_\lambda$ of λ in $K_{\mathbb{C}}$ is

$$(K_{\mathbb{C}})_\lambda = \left\{ k = \begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix} \mid g_1 \in O(m, \mathbb{C}), g_2 \in GL(n-m, \mathbb{C}) \right\}. \tag{7.64}$$

We denote the determinant of the isotropy representation by $\det(\operatorname{Ad}|_{T_\lambda \mathcal{O}_m}) : (K_{\mathbb{C}})_\lambda \to \mathbb{C}^\times$, where $T_\lambda \mathcal{O}_m$ is the tangent space of $\mathcal{O}_m$ at λ. It is written by

$$\det(\operatorname{Ad}|_{T_\lambda \mathcal{O}_m}) = (\det g_1)^{n-m} (\det g_2)^{-m} = (\det g_1)^n (\det k)^{-m},$$

with the notation (7.64). The cotangent bundle $T^*\mathcal{O}_m$ is a $K_{\mathbb{C}}$-equivariant vector bundle. The line bundle $\Lambda^{\text{top}} = \bigwedge^{\dim \mathcal{O}_m} T^*\mathcal{O}_m$ consisting of volume forms on the orbit $\mathcal{O}_m$ is a $K_{\mathbb{C}}$-equivariant line bundle. Then it corresponds to the one-dimensional representation of the isotropy subgroup $(K_{\mathbb{C}})_\lambda$. In this case it is given by the coisotropy representation

$$\det(\operatorname{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) : (K_{\mathbb{C}})_\lambda \ni k \mapsto (\det g_1)^{-n} (\det k)^m \in \mathbb{C}^\times,$$

with the notation (7.64). We introduce the square root of the line bundle Λ^{top}, denoted by ξ, and consider the set $\Gamma(\mathcal{O}_m, \xi)$ of its global sections. We will give the relation between this line bundle on the orbit $\mathcal{O}_m$ and the lowest weight representation under consideration.

In what follows, *we assume that n is even.* We define the one-dimensional representation

$$\xi : (\widetilde{K_{\mathbb{C}}})_\lambda \ni k \mapsto \det{}^{m/2} k \in \mathbb{C}^\times .$$

By the definition, the coisotropy representation is the square of ξ;

$$\det(\mathrm{Ad}^*\big|_{T^*_\lambda \mathcal{O}_m}) = \det(\mathrm{Ad}\big|_{T_\lambda \mathcal{O}_m})^{-1} = \xi^2.$$

This means that ξ corresponds to the half-form bundle on the orbit $\mathcal{O}_m = K_{\mathbb{C}}/(K_{\mathbb{C}})_\lambda = \widetilde{K_{\mathbb{C}}}/(\widetilde{K_{\mathbb{C}}})_\lambda$. The set of global sections $\Gamma(\mathcal{O}_m, \xi)$ has a natural $\widetilde{K_{\mathbb{C}}}$-module structure.

Proposition 7.6. — *For $0 \le m < n$ and $n \in 2\mathbb{Z}$, the lowest weight module $L(\mathbf{1}_{O(m)})$ is isomorphic to $\Gamma(\mathcal{O}_m, \xi)$ as $\widetilde{K}$-modules.*

Proof. — We denote the complexification of the character $\chi_1 : \widetilde{K} \to \mathbb{C}^\times$ introduced in Section 6.1 by the same character. To be more explicit, we define the character $\chi_1 : \widetilde{K_{\mathbb{C}}} \to \mathbb{C}^\times$ by $\chi_1(k) = \det{}^{m/2} k$. The restriction of χ_1 to the isotropy subgroup coincides with ξ. Then, we see that

$$\Gamma(\mathcal{O}_m, \xi) = \mathrm{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi = \chi_1 \otimes \mathrm{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \mathbf{1}_{(\widetilde{K_{\mathbb{C}}})_\lambda} = \chi_1 \otimes \mathbb{C}[\mathcal{O}_m]$$

for $0 \le m \le n$ as $\widetilde{K_{\mathbb{C}}}$-modules. On the other hand, we have seen in Corollary 7.3(2) that

$$L(\mathbf{1}_{O(m)}) = \mathbb{C}[\overline{\mathcal{O}_m}] \otimes \det{}^{m/2}.$$

Since $\overline{\mathcal{O}_m}$ is normal (cf. Lemma 7.1), and, for $m \ne n$, $\operatorname{codim}_{\overline{\mathcal{O}_m}} \mathcal{O}_m \ge 2$ for $m \ne n$, the restriction map gives a natural isomorphism $\mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathcal{O}_m]$ (cf. [**10**, Chapter 11, § 11.2]). This shows the proposition. $\square$

7.4. The case $G = U(p,q)$. — Let $G = U(p,q)$. In this case, $K = U(p) \times U(q)$, $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$, $\mathfrak{p}^- = M(q,p,\mathbb{C})$. The action of $(k_1, k_2) \in K_{\mathbb{C}}$ on $A \in \mathfrak{p}^-$ is given by

$$k_2 A k_1^{-1}. \tag{7.65}$$

Put $r = \mathbb{R}\text{-rank}\, U(p,q) = \min(p,q)$. We define a subset of $M_{q,p} = M(q,p,\mathbb{C})$ by

$$\mathcal{O}_m = \{A \in M_{q,p} \mid \operatorname{rank}(A) = m\}, \qquad (m = 0, 1, \dots, r).$$

By an argument similar to the case $Sp(2n,\mathbb{R})$, we know that $\mathcal{O}_m$ is a $K_{\mathbb{C}}$-orbit, and they give a complete classification of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$. Note that the matrix $\sum_{j \le m} E_{jj}$ is contained in $\mathcal{O}_m$. It is easy to see that

$$\dim \mathcal{O}_m = (p+q)m - m^2,$$

hence all the orbits have different dimensions. The defining ideal I_m of $\overline{\mathcal{O}_m}$ is generated by the minors

$$\{D_{IJ} \mid I \subset \{1, 2, \dots, q\}, J \subset \{1, 2, \dots, p\}, |I| = |J| = m+1\}.$$

The affine algebraic variety $\overline{\mathcal{O}_m}$ is called the determinantal variety.

Now recall the dual pair $(U(p,q), U(m))$. Let $GL(m,\mathbb{C})$ be the complexification of $U(m)$. We consider the natural action of $K_{\mathbb{C}} \times GL(m,\mathbb{C}) = (GL(p,\mathbb{C}) \times GL(q,\mathbb{C})) \times GL(m,\mathbb{C}) \ni (k_1, k_2, h)$ on $(A, B) \in M_{m,p} \oplus M_{m,q} \simeq M_{m,p+q}$ by

$$({}^t h^{-1} A k_1^{-1}, hB\, {}^t k_2), \tag{7.66}$$

which comes from (6.46). For $(A, B) \in M_{m,p} \oplus M_{m,q}$, we define an unfolding map ψ by

$$\psi(A, B) = {}^t BA \in M_{q,p}.$$

This is a polynomial map of degree two. Note that $\psi(\sum_{l \le j} E_{ll}, \sum_{l \le j} E_{ll}) = \sum_{l \le j} E_{ll} \in \mathcal{O}_m$. From this, we see that the image of ψ coincides with $\overline{\mathcal{O}_m}$. With the trivial action of $GL(m,\mathbb{C})$ on $\overline{\mathcal{O}_m}$, the map

$$\psi : M_{m,p+q} \to \overline{\mathcal{O}_m}$$

is $K_{\mathbb{C}} \times GL(m,\mathbb{C})$-equivariant, that is, $\psi({}^t h^{-1} A k_1^{-1}, hB\, {}^t k_2) = k_2 \psi(A, B) k_1^{-1}$ for all $(k_1, k_2) \in K_{\mathbb{C}}$, $h \in GL(m,\mathbb{C})$. This map induces the $\mathbb{C}$-algebra homomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \ni f \mapsto f \circ \psi \in \mathbb{C}[M_{m,p+q}]^{GL(m,\mathbb{C})}.$$

As a summary we have

Lemma 7.7. — *There exists a $\mathbb{C}$-algebra isomorphism*

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \to \mathbb{C}[M_{m,p+q}]^{GL(m,\mathbb{C})} = S(M_{p+q,m})^{GL(m,\mathbb{C})},$$

which means that $\overline{\mathcal{O}_m}$ is the geometric quotient $M_{p+q,m}//GL(m,\mathbb{C})$. In particular, $\overline{\mathcal{O}_m}$ is a normal variety. Here we consider $M_{p+q,m}$ as the contragredient space to $M_{m,p+q}$.

Proof. — It is injective since ψ is surjective. The classical invariant theory also says that every $GL(m,\mathbb{C})$-invariants on $M_{m,p+q}$ is generated by typical invariants of degree two, that is, this map ψ^* is surjective. $\square$

For the Weil representation of the dual pair $(U(p,q), U(m)) \in Sp(2nm,\mathbb{R})$ and the unitary lowest weight module $L(\mathbf{1}_{U(m)})$, we have expected the same story. Take a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ consisting of diagonal matrices

$$\mathfrak{t} = \{H = \operatorname{diag}(a_1, \ldots, a_p, b_1, \ldots, b_q) \mid a_i, b_j \in \mathbb{C}\}.$$

This is also a Cartan subalgebra of $\mathfrak{g}$. We define $\varepsilon_i, \delta_j \in \mathfrak{t}^*$ by $\varepsilon_i(H) = a_i, \delta_j(H) = b_j$ for above $H \in \mathfrak{t}$. Then the set of positive non-compact roots is

$$\Delta_n^+ = \{\varepsilon_i - \delta_j \mid 1 \le i \le p, 1 \le j \le q\}.$$

Put

$$X_{\varepsilon_a - \delta_b} = \left(\begin{array}{c|c} 0 & E_{ab} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{gl}(p+q,\mathbb{C}) = \mathfrak{g}.$$

Then $X_{\varepsilon_a-\delta_b}$ is a non-compact root vector in $\mathfrak{p}^+$. From the embedding (3.19) and the Fock realization (4.27) of the Weil representation Ω, we conclude that

$$\Omega(-2X_{\varepsilon_a-\delta_b}) = \psi_{ab} = \sum_{j=1}^{m} x_{aj}y_{bj} \qquad (1 \leq a \leq p, 1 \leq b \leq q), \tag{7.67}$$

where $(x_{aj})_{1\leq a\leq p,1\leq j\leq m} \in M_{p,m}$ and $(y_{bj})_{1\leq b\leq q,1\leq j\leq m} \in M_{q,m}$. Note that these quadratics (7.67) generate the full invariants $S(M_{p+q,m})^{GL(m,\mathbb{C})}$. From this, we get

Lemma 7.8. — *There are algebra isomorphisms*

$$\Omega(U(\mathfrak{p}^+)) \simeq \mathbb{C}[M_{m,p+q}]^{GL(m,\mathbb{C})} \simeq \mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathfrak{p}^-]/I_m.$$

We have $\operatorname{Ann} L(\mathbf{1}_{U(m)}) = \operatorname{Ann} \operatorname{gr} L(\mathbf{1}_{U(m)}) = I_m$ *in* $U(\mathfrak{p}^+)$.

Proof. — The proof is similar to that of Lemma 7.2. □

Corollary 7.9. — (1) *The associated variety of* $L(\mathbf{1}_{U(m)})$ *is* $\overline{\mathcal{O}_m}$.
(2) *As a* $\widetilde{K}$*-module,* $L(\mathbf{1}_{U(m)})$ *is isomorphic to* $\mathbb{C}[\overline{\mathcal{O}_m}] \otimes (\det{}^{m/2} \boxtimes \det{}^{-m/2})$.
(3) *Bernstein degree of* $L(\mathbf{1}_{U(m)})$ *coincides with* $\deg \overline{\mathcal{O}_m}$.

Proof. — The proof is similar to that of Corollary 7.3. For the K-type decomposition of $L(\mathbf{1}_{U(m)})$, see (6.54). □

Let us define the natural filtration of $L = L(\mathbf{1}_{U(m)})$ by $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$, where $\mathbf{1}$ is a constant polynomial. By (6.55), we know

$$\begin{aligned}
\dim L_k &= \sum_{\substack{\lambda\in\mathcal{P}_m \\ |\lambda|\leq k}} \dim \tau_\lambda^{(p)} \dim \tau_\lambda^{(q)} \\
&= \sum_{\substack{\lambda, l(\lambda)\leq m \\ |\lambda|\leq k}} \frac{\prod_{1\leq i<j\leq m}(\lambda_i-\lambda_j-i+j) \prod_{\substack{1\leq i\leq m, \\ m+1\leq j\leq p}}(\lambda_i-i+j) \prod_{m+1\leq i<j\leq p}(j-i)}{\prod_{1\leq i<j\leq p}(j-i)} \\
&\qquad \times \frac{\prod_{1\leq i<j\leq m}(\lambda_i-\lambda_j-i+j) \prod_{\substack{1\leq i\leq m, \\ m+1\leq j\leq q}}(\lambda_i-i+j) \prod_{m+1\leq i<j\leq q}(j-i)}{\prod_{1\leq i<j\leq q}(j-i)} \\
&= \frac{k^{m(m-1)/2\times 2+m(p+q-2m)+m}}{\prod_{i=1}^{m}(p-i)!(q-i)!} \\
&\qquad \times \int_{\substack{0\leq x_m\leq x_{m-1}\leq\cdots\leq x_1, \\ x_1+\cdots+x_m\leq 1}} \prod_{1\leq i<j\leq m}(x_i-x_j)^2 \prod_{i=1}^{m} x_i^{p+q-2m} dx_1\ldots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k)
\end{aligned}$$

$$= \frac{k^{m(p+q-m)}}{m! \prod_{i=1}^{m} (p-i)!(q-i)!} \int_{\substack{0 \le x_i \\ x_1 + \cdots + x_m \le 1}} \prod_{1 \le i < j \le m} |x_i - x_j|^2 \prod_{i=1}^{m} x_i^{p+q-2m} dx_1 \ldots dx_m$$
$$+ \text{(lower order terms of } k)$$

for sufficiently large k.

Theorem 7.10. — *Assume that $m \le \min(p,q) = \mathbb{R}\text{-rank}\, U(p,q)$, and consider the reductive dual pair $(U(p,q), U(m))$.*
(1) *The unitarizable lowest weight module $L(\mathbf{1}_{U(m)})$ of $U(p,q)^\sim$ has the lowest weight $m/2\mathbb{I}_{p,q} = m/2 \left(\sum_{i=1}^{p} \varepsilon_i - \sum_{j=1}^{q} \delta_j \right)$, where $\mathbb{I}_{p,q} = (1, \ldots, 1, -1, \ldots, -1)$. Its associated cycle is given by $\mathcal{AC}\,(L(\mathbf{1}_{U(m)})) = [\overline{\mathcal{O}_m}]$.*
(2) *The Gelfand-Kirillov dimension and the Bernstein degree of $L(\mathbf{1}_{U(m)})$ is given by*

$$\begin{aligned} \operatorname{Dim} L(\mathbf{1}_{U(m)}) &= \dim \overline{\mathcal{O}_m} = m(p+q-m), \\ \operatorname{Deg} L(\mathbf{1}_{U(m)}) &= \deg \overline{\mathcal{O}_m} = \prod_{j=1}^{m} \frac{(j-1)!\,(p+q-m-j)!}{(p-j)!\,(q-j)!} \end{aligned}$$

Proof. — By the formula of $\dim L_k$ above, we have

$$\begin{aligned} \operatorname{Dim} L(\mathbf{1}_{U(m)}) &= m(p+q-m) =: d, \\ \operatorname{Deg} L(\mathbf{1}_{U(m)}) &= \frac{d!}{m! \prod_{i=1}^{m} (p-i)!(q-i)!} I^2(p+q-2m, m). \end{aligned}$$

Now apply Theorem 7.4. □

We show that the half-form bundle on $\mathcal{O}_m$ is related to some lowest weight representation $L(\sigma)$. We put

$$\lambda = \sum_{j=1}^{m} E_{jj} \in \mathcal{O}_m \subset M_{q,p} \cong \mathfrak{p}^-.$$

The group $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ acts on $\mathcal{O}_m$ by (7.65). The stabilizer $(K_{\mathbb{C}})_\lambda$ of λ in $K_{\mathbb{C}}$ is

$$(K_{\mathbb{C}})_\lambda = \left\{ (k_1, k_2) = \left(\begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix}, \begin{pmatrix} g_1 & * \\ 0 & g_3 \end{pmatrix} \right) \in K_{\mathbb{C}} \mid g_1 \in GL(m,\mathbb{C}) \right\}. \quad (7.68)$$

The determinant of the isotropy representation is

$$\det(\operatorname{Ad}|_{T_\lambda \mathcal{O}_m}) = (\det g_1)^{p-q} (\det k_1)^{-m} (\det k_2)^m,$$

and that of the coisotropy representation

$$\det(\operatorname{Ad}^*|_{T_\lambda^* \mathcal{O}_m}) : (K_{\mathbb{C}})_\lambda \ni (k_1, k_2) \mapsto (\det g_1)^{-(p-q)} (\det k_1)^m (\det k_2)^{-m} \in \mathbb{C}^\times,$$

with the notation (7.68). We denote the line bundle consisting of volume forms on $\mathcal{O}_m$ by Λ^{top}, and its square root by ξ. Let us clarify the meaning of the square root ξ of Λ^{top}. We denote the inverse image of the subgroup $K_{\mathbb{C}} \subset \mathcal{K}_{\mathbb{C}}$ in $\widetilde{\mathcal{K}_{\mathbb{C}}}$ by $\widetilde{K}_{\mathbb{C}}$. This is a double covering group of $K_{\mathbb{C}}$, which is not necessarily connected, with the covering map $\widetilde{K_{\mathbb{C}}} \longrightarrow K_{\mathbb{C}}$. We have an realization

$$\widetilde{K_{\mathbb{C}}} = \{(k,z) \in K_{\mathbb{C}} \times \mathbb{C}^{\times} \mid k = (k_1,k_2), (\det k_1)^m (\det k_2)^{-m} = z^2\}.$$

Through the natural projection, $\widetilde{K_{\mathbb{C}}}$ also acts on $\mathcal{O}_m$. We denote the isotropy subgroup at $\lambda \in \mathcal{O}_m$ by $(\widetilde{K_{\mathbb{C}}})_\lambda$. This is the inverse image of $(K_{\mathbb{C}})_\lambda$, that is,

$$(\widetilde{K_{\mathbb{C}}})_\lambda = \left\{ ((k_1,k_2),z) \in \widetilde{K_{\mathbb{C}}} \mid k_1 = \begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix}, k_2 = \begin{pmatrix} g_1 & * \\ 0 & g_3 \end{pmatrix}, g_1 \in GL(m,\mathbb{C}) \right\}. \tag{7.69}$$

In what follows, *we assume that* $p-q$ *is even* and calculate the K-types of $\Gamma(\mathcal{O}_m,\xi)$. There exists a well-defined character

$$\xi : (\widetilde{K_{\mathbb{C}}})_\lambda \ni (k_1,k_2,z) \mapsto (\det g_1)^{-(p-q)/2} z \in \mathbb{C}^{\times},$$

with the notation (7.69). By the construction of ξ, the coisotropy representation is the square of ξ:

$$\det(\mathrm{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) = \xi^2.$$

This means that ξ determines the half-form bundle $\sqrt{\Lambda^{\mathrm{top}}}$ on the orbit $\mathcal{O}_m$. As a $\widetilde{K_{\mathbb{C}}}$-module, the set of global sections $\Gamma(\mathcal{O}_m,\xi)$ is isomorphic to the induced module

$$\Gamma(\mathcal{O}_m,\xi) = \mathrm{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi.$$

We define a character $\chi_1 : \widetilde{K_{\mathbb{C}}} \longrightarrow \mathbb{C}^{\times}$ by $\chi_1(k_1,k_2,z) = z$, and $\xi' : (K_{\mathbb{C}})_\lambda \to \mathbb{C}^{\times}$ by $\xi'(k_1,k_2) = (\det g_1)^{-(p-q)/2}$ in the notation above. The character ξ' lifts up to a character of $(\widetilde{K_{\mathbb{C}}})_\lambda$ via projection map, and we denote it by the same letter ξ' again. Roughly speaking, χ_1 equals "$\det^{m/2} k_1 \det^{-m/2} k_2$". Then, ξ is the tensor product of ξ' with the restriction of χ_1 to the subgroup $(\widetilde{K_{\mathbb{C}}})_\lambda$. By the reciprocity law,

$$\mathrm{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi = \chi_1 \otimes \mathrm{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi' = \chi_1 \otimes \mathrm{Ind}_{(K_{\mathbb{C}})_\lambda}^{K_{\mathbb{C}}} \xi'.$$

Lemma 7.11. — *We assume that* $m \le \min(p,q)$ *and* $p-q \in 2\mathbb{Z}$ *as before, and that* $\max(p,q) \neq m$. *Then, as a* $K_{\mathbb{C}}$*-module, we have an isomorphism*

$$\mathrm{Ind}_{(K_{\mathbb{C}})_\lambda}^{K_{\mathbb{C}}} \xi' = \sum_{\lambda \in \mathcal{P}_m}^{\oplus} \tau_{\lambda + l\mathbb{I}_m} \boxtimes \tau_\lambda^*$$

with $l = (q-p)/2$. *Here we denote* $\mathbb{I}_m = (1,\dots,1,0,\dots,0)$, *in which* 1 *appears* m*-times.*

This shows that

$$\Gamma(\mathcal{O}_m,\xi) = \sum_{\lambda \in \mathcal{P}_m}^{\oplus} (\tau_{\lambda + l\mathbb{I}_m} \otimes \det{}^{m/2}) \boxtimes (\tau_\lambda \otimes \det{}^{m/2})^*.$$

On the other hand, by (6.50), the lowest weight module $L(\chi^{-1})$ also has the same $\widetilde{K}$-types. Indeed, the character χ of $G_2 = U(m)$ is $\det^{(p-q)/2} = \det^{-l}$ as is shown in (6.48). For $\nu = l\mathbb{I}_m$, we see that the multiplicity $m(\lambda,\mu;\nu)$ defined by (6.49) is

$$m(\lambda,\mu;\nu) = \begin{cases} 1 & \text{if } \lambda = \mu + \nu \\ 0 & \text{otherwise.} \end{cases}$$

Summarizing above, we have

Proposition 7.12. — *Suppose $p - q \in 2\mathbb{Z}$ and $0 \le m < \min(p,q)$. Let $\chi = \det^{(p-q)/2}$ be the character of $G_2 = U(m)$ given in* (6.48). *Then the lowest weight module $L(\chi^{-1})$ is isomorphic to $\Gamma(\mathcal{O}_m, \xi)$ as a $\widetilde{K_{\mathbb{C}}}$-module.*

7.5. The case $G = O^*(2p)$. — Let us consider the case $G = O^*(2p)$. In this case $K = U(p)$, $K_{\mathbb{C}} = GL(p,\mathbb{C})$, $\mathfrak{p}^- = \mathrm{Alt}\,(p,\mathbb{C})$. The action of $k \in K_{\mathbb{C}}$ on $A \in \mathfrak{p}^-$ is given by ${}^t k^{-1} A k^{-1}$.

Put $r = \mathbb{R}\text{-rank}\, O^*(2p) = [p/2]$, where $[x]$ is the Gauss symbol. We define a subset of $\mathrm{Alt}\,(p,\mathbb{C})$ by

$$\mathcal{O}_m = \{A \in \mathrm{Alt}\,(p,\mathbb{C}) \mid \mathrm{rank}\,(A) = 2m\}, \qquad (m = 0, 1, \ldots, r).$$

Since the rank of alternative matrices is always even, these $\{\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r\}$ form the set of all $K_{\mathbb{C}}$-orbits on $\mathrm{Alt}\,(p,\mathbb{C})$. The matrix $\sum_{j=1}^{m}(E_{m+j,j} - E_{j,m+j})$ is contained in $\mathcal{O}_m$. The dimension of the orbit is given by

$$\dim \mathcal{O}_m = 2pm - m(2m+1),$$

and the defining ideal I_m of $\overline{\mathcal{O}_m}$ is generated by

$$\{D_{IJ} \mid I, J \subset \{1, 2, \ldots, 2p\}, |I| = |J| = 2m+1\}.$$

Recall the dual pair $(O^*(2p), Sp(2m))$. Let $Sp(2m,\mathbb{C})$ be the complexification of $Sp(2m)$. We define the action of $K_{\mathbb{C}} \times Sp(2m,\mathbb{C})$ on $A \in M_{2m,p}$ by

$$(k,h) \cdot A = hAk^{-1}, \qquad \text{for } k \in GL(p,\mathbb{C}) = K_{\mathbb{C}},\ h \in Sp(2m,\mathbb{C}).$$

We define an unfolding map ψ by

$$\psi(A) = {}^t A J_m A \qquad \text{for } A \in M_{2m,p},$$

where J_m is defined as in (3.13). This is a polynomial map of degree two. Since

$$\psi(\sum_{j \le 2m} E_{jj}) = \sum_{j=1}^{m}(E_{m+j,j} - E_{j,m+j}) \in \mathcal{O}_m,$$

we see that the image of ψ coincides with $\overline{\mathcal{O}_m}$. With the trivial action of $Sp(2m,\mathbb{C})$ on $\overline{\mathcal{O}_m}$, the map

$$\psi : M_{2m,p} \to \overline{\mathcal{O}_m}$$

is $K_{\mathbb{C}} \times Sp(2m,\mathbb{C})$-equivariant, that is, $\psi(hAk^{-1}) = {}^t k^{-1}\psi(A)k^{-1}$ for all $k \in K_{\mathbb{C}}$ and $h \in Sp(2m,\mathbb{C})$. This map induces a $\mathbb{C}$-algebra homomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \ni f \mapsto f \circ \psi \in \mathbb{C}[M_{2m,p}]^{Sp(2m,\mathbb{C})}.$$

Lemma 7.13. — *We have a* $\mathbb{C}$*-algebra isomorphism*

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \to \mathbb{C}[M_{2m,p}]^{Sp(2m,\mathbb{C})},$$

which means that $\overline{\mathcal{O}_m}$ *is the geometric quotient* $M_{2m,p}//Sp(2m,\mathbb{C})$*. In particular,* $\overline{\mathcal{O}_m}$ *is a normal variety.*

Proof. — The proof is similar to that of Lemma 7.1. □

Let us consider the Weil representation of the dual pair $(O^*(2p), Sp(2m)) \in Sp(2nm,\mathbb{R})$ $(n = 2p)$ and the unitary lowest weight module $L(\mathbf{1}_{Sp(2m)})$. Take a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ consisting of diagonal matrices

$$\mathfrak{t} = \{H = \operatorname{diag}(a_1, \dots, a_p, -a_1, \dots, -a_p) \mid a_i \in \mathbb{C}\}.$$

This is also a Cartan subalgebra of $\mathfrak{g}$. We define $\varepsilon_i \in \mathfrak{t}^*$ by $\varepsilon_i(H) = a_i$ for above $H \in \mathfrak{t}$. Then the set of positive non-compact roots is

$$\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \le i < j \le p\}.$$

Put

$$X_{\varepsilon_a+\varepsilon_b} = \left(\begin{array}{c|c} 0 & E_{ab} - E_{ba} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{o}(2p,\mathbb{C}) = \mathfrak{g}.$$

Note that the complexification $\mathfrak{o}(2p,\mathbb{C})$ is given in (7.60), in which we adopt rather non-standard symmetric bilinear form S_p. Then $X_{\varepsilon_a+\varepsilon_b}$ is a non-compact root vector in $\mathfrak{p}^+$. From the embedding (3.22) and the Fock realization (4.27) of Ω, we get

$$\Omega(-2X_{\varepsilon_a+\varepsilon_b}) = \psi_{ab} = \sum_{j=1}^{m}(x_{aj}y_{bj} - x_{bj}y_{aj}) \qquad (1 \le a < b \le p), \tag{7.70}$$

where $\left((x_{aj})_{1\le a\le p, 1\le j\le m}, (y_{bj})_{1\le b\le p, 1\le j\le m}\right) \in M_{p,2m} = M^*_{2m,p}$. These quadratics (7.70) generate the invariants $S(M_{p,2m})^{Sp(2m,\mathbb{C})}$.

Lemma 7.14. — *There are algebra isomorphisms*

$$\Omega(U(\mathfrak{p}^+)) \simeq \mathbb{C}[M_{2m,p}]^{Sp(2m,\mathbb{C})} \simeq \mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathfrak{p}^-]/I_m.$$

We have $\operatorname{Ann} L(\mathbf{1}_{Sp(2m)}) = \operatorname{Ann}\operatorname{gr} L(\mathbf{1}_{Sp(2m)}) = I_m$ *in* $U(\mathfrak{p}^+)$.

Proof. — The proof is similar to that of Lemma 7.2. □

Corollary 7.15. — (1) *The associated variety of* $L(\mathbf{1}_{Sp(2m)})$ *is* $\overline{\mathcal{O}_m}$.
(2) *As a* K*-module,* $\mathbb{C}[\overline{\mathcal{O}_m}]$ *is isomorphic to* $L(\mathbf{1}_{Sp(2m)})$.
(3) *The Bernstein degree of* $L(\mathbf{1}_{Sp(2m)})$ *coincides with* $\deg\overline{\mathcal{O}_m}$.

Proof. — The proof is similar to that of Corollary 7.3. For the K-type decomposition of $L(\mathbf{1}_{Sp(2m)})$, see (6.58). □

Let us define the natural filtration of $L = L(\mathbf{1}_{Sp(2m)})$ by $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$, where $\mathbf{1}$ is the constant polynomial. By (6.59), we know

$$\dim L_k = \sum_{\substack{\lambda\in\mathcal{P}_m \\ |\lambda|\le k}} \dim \tau^{(p)}_{\lambda^\#}$$

$$
\begin{aligned}
&= \sum_{\substack{|\lambda|\le k\\ l(\lambda)\le m}} \frac{\prod_{1\le i<j\le m} (\lambda_i-\lambda_j-2i+2j)^2((\lambda_i-\lambda_j-2i+2j)^2-1)}{\prod_{1\le i<j\le p}(j-i)} \\
&\quad\times \prod_{\substack{1\le i\le m\\ 2m+1\le j\le p}} (\lambda_i-2i+j)(\lambda_i-2i+1+j) \prod_{2m+1\le i<j\le p}(j-i) \\
&= \frac{k^{m(m-1)/2\times 4+2m(p-2m)+m}}{\prod_{i=1}^{2m}(p-i)!} \\
&\quad\times \int_{\substack{0\le x_m\le\cdots\le x_1\\ x_1+\cdots+x_m\le 1}} \prod_{1\le i<j\le m}(x_i-x_j)^4 \prod_{i=1}^m x_i^{2(p-2m)}\,dx_1\cdots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k) \\
&= \frac{k^{m(2p-2m-1)}}{m!\prod_{i=1}^{2m}(p-i)!} \int_{\substack{0\le x_i\\ x_1+\cdots+x_m\le 1}} \prod_{1\le i<j\le m}|x_i-x_j|^4 \prod_{i=1}^m x_i^{2(p-2m)}\,dx_1\cdots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k)
\end{aligned}
$$

for sufficiently large k.

Theorem 7.16. — *Assume that $m \le [p/2] = \mathbb{R}\text{-rank}\, O^*(2p)$, and consider the reductive dual pair $(O^*(2p), Sp(2m))$.*

(1) *The unitarizable lowest weight module $L(\mathbf{1}_{Sp(2m)})$ of $O^*(2p)^\sim$ has the lowest weight $m(1,\dots,1) = m\sum_{i=1}^p \varepsilon_i$, and its associated cycle is given by $\mathcal{AC}\,(L(\mathbf{1}_{Sp(2m)})) = [\overline{\mathcal{O}_m}]$.*

(2) *The Gelfand-Kirillov dimension and the Bernstein degree of $L(\mathbf{1}_{Sp(2m)})$ is given by*

$$
\begin{aligned}
\operatorname{Dim} L(\mathbf{1}_{Sp(2m)}) &= \dim \overline{\mathcal{O}_m} = m(2p-2m-1), \\
\operatorname{Deg} L(\mathbf{1}_{Sp(2m)}) &= \deg \overline{\mathcal{O}_m} = (2m-1)!! \prod_{j=1}^m \frac{\big(2(j-1)\big)!\,\big(2(p-m-j)\big)!}{(p-j)!\,(p-m-j)!}.
\end{aligned}
$$

Proof. — By the formula of $\dim L_k$, we get

$$
\begin{aligned}
\operatorname{Dim} L(\mathbf{1}_{Sp(2m)}) &= m(2p-2m-1) =: d, \\
\operatorname{Deg} L(\mathbf{1}_{Sp(2m)}) &= \frac{d!}{m!\prod_{i=1}^{2m}(p-i)!} I^4(2p-4m, m).
\end{aligned}
$$

Apply Theorem 7.4 to get the desired formula. □

We have a relation between the half-form bundle and $L(\mathbf{1}_{Sp(2m)})$ similar to that in Proposition 7.6. We define $\lambda = \begin{pmatrix} J_m & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{Alt}(p,\mathbb{C})$ with $J_m = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix}$.

The isotropy subgroup of λ in $K_{\mathbb{C}}$ is

$$(K_{\mathbb{C}})_\lambda = \left\{ k = \begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix} \mid g_1 \in Sp(2m, \mathbb{C}), g_2 \in GL(p-2m, \mathbb{C}) \right\}.$$

Therefore the determinant of the coisotropy representation becomes

$$\det(\mathrm{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) = (\det k)^{2m},$$

and we define its square root by

$$\xi : (K_{\mathbb{C}})_\lambda \ni k \mapsto (\det k)^m \in \mathbb{C}^\times.$$

Proposition 7.17. — *For $m < p/2$, the set of global section of the half-form bundle $\Gamma(\mathcal{O}_m, \xi)$ is isomorphic to $L(\mathbf{1}_{Sp(2m)})$ as a $\widetilde{K}$-module.*

7.6. A unified formula. — Consider the reductive dual pair $(G_1, G_2) \subset \mathcal{G} = Sp(2nm, \mathbb{R})$ of compact type. We put $G = G_1$, which is a non-compact companion. We use the notation in § 3 freely in this subsection. In particular, $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is a division algebra over $\mathbb{R}$, and $n = 1/2 \dim_{\mathbb{R}} V_1, m = \dim_D V_2$. Put $r = \mathbb{R}\text{-rank}\, G$, and $\alpha = \dim_{\mathbb{R}} D = 1, 2, 4$.

Summarizing the above three explicit calculations, we have a unified expression of the Gelfand-Kirillov dimension and the Bernstein degree of the unitary lowest weight module $L(\mathbf{1}_{G_2})$.

Theorem 7.18. — *Assume that the dual pair (G_1, G_2) is in the stable range, i.e., $m \le r$. We denote by $L(\mathbf{1}_{G_2})$ the irreducible lowest weight module of $\widetilde{G_1}$ which is the (twisted) theta lift of the trivial representation of the compact companion G_2. Then the associated cycle $\mathcal{AC}\, L(\mathbf{1}_{G_2})$ is the closure of the m-th $K_{\mathbb{C}}$-orbit $\overline{\mathcal{O}_m}$ in $\mathfrak{p}^-$. Moreover, we have*

$$\mathrm{Dim}\, L(\mathbf{1}_{G_2}) = m\Big(n + 1 - \frac{\alpha}{2}(m+1)\Big) = \dim \overline{\mathcal{O}_m} =: d,$$

and

$$\mathrm{Deg}\, L(\mathbf{1}_{G_2}) = F^{-1} \frac{d!}{m!} I^\alpha(n - \alpha m, m) = \deg \overline{\mathcal{O}_m},$$

where $I^\alpha(s, m)$ is the integral (7.63), *and the integer F is given by*

$$F = \begin{cases} \prod_{j=1}^m (2(n-j))!! = 2^{m-d} \prod_{j=1}^m (n-j)! & \text{Case } (Sp, O), \\ \prod_{j=1}^m (p-j)!(q-j)! & \text{Case } (U, U), \\ \prod_{j=1}^{2m} (p-j)! & \text{Case } (O^*, Sp). \end{cases}$$

Remark 7.19. — If G/K is of tube type, we have

$$F = \begin{cases} \prod_{j=1}^m (2(n-j))!! = 2^{m-d} \prod_{j=1}^m (n-j)! & \text{Case } (Sp, O), \\ \prod_{j=1}^m \{(n/\alpha - j)!\}^2 & \text{Case } (U, U), \\ \prod_{j=1}^{2m} (n/\alpha - j)! & \text{Case } (O^*, Sp). \end{cases}$$

8. Multiplicity free action and Poincaré series

In this section, we develop a general theory on Poincaré series of graded modules which arise from multiplicity free action of reductive groups. All the groups in this section are complex algebraic groups and irreducible representations are finite dimensional ones.

8.1. Poincaré series of covariants. — Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be complex reductive groups and X a vector space on which $\mathcal{G}_1$ and $\mathcal{G}_2$ jointly act linearly. We assume that the action of $\mathcal{G}_1 \times \mathcal{G}_2$ is multiplicity free. This means that the polynomial ring $\mathbb{C}[X] = \Gamma(X)$ decomposes, as a $\mathcal{G}_1 \times \mathcal{G}_2$-module, into irreducible representations with multiplicity one. Namely, there exists a subset $R_X(\mathcal{G}_1 \times \mathcal{G}_2) \subset \mathrm{Irr}(\mathcal{G}_1 \times \mathcal{G}_2)$ such that

$$\Gamma(X) = \sum_{\pi_1 \boxtimes \pi_2 \in R_X(\mathcal{G}_1 \times \mathcal{G}_2)}^{\oplus} \pi_1 \boxtimes \pi_2.$$

We assume further, in the decomposition, the correspondence $\pi_1 \leftrightarrow \pi_2$ is one to one. Hence, π_1 determines π_2 and vice versa.

We choose suitable positive systems of roots for $\mathcal{G}_1$ and $\mathcal{G}_2$, and fix them in what follows. Let λ be the highest weight of $\pi_1 = \pi_1(\lambda)$ with respect to the positive system we chose. Then we will denote the corresponding highest weight of π_2 by $\varphi(\lambda)$ so that $\pi_2 = \pi_2(\varphi(\lambda))$. Let Λ^+ be a lattice semigroup of the highest weights of $\pi_1 \in \mathrm{Irr}(\mathcal{G}_1)$ which occur in $\Gamma(X)$. Then we can write the decomposition as

$$\Gamma(X) = \sum_{\lambda \in \Lambda^+}^{\oplus} \pi_1(\lambda) \boxtimes \pi_2(\varphi(\lambda)).$$

Note that the correspondence $\Lambda^+ \ni \lambda \mapsto \varphi(\lambda)$ is a semigroup morphism from Λ^+ into the dominant weight lattice of $\mathcal{G}_2$, i.e., $\varphi(\lambda + \eta) = \varphi(\lambda) + \varphi(\eta)$.

We consider a (reductive) spherical subgroup H of $\mathcal{G}_1$. Since H is spherical, for any irreducible representation (π_1, V) of $\mathcal{G}_1$, V has at most one-dimensional invariants under the action of H : $\dim V^H \leq 1$. We put

$$\Lambda^+(H) = \{\lambda \in \Lambda^+ \mid \dim V_\lambda^H = 1\} \subset \Lambda^+,$$

where V_λ is a representation space of $\pi_1(\lambda)$. Let Λ (respectively $\Lambda(H)$) be the lattice generated by Λ^+ (respectively $\Lambda^+(H)$). Note that it is not necessary to hold that $\Lambda^+(H) = \Lambda^+ \cap \Lambda(H)$. Since Λ^+ is a free abelian semigroup generated by finite elements (see the argument in [**26**, §2]), we can extend the correspondence $\varphi(\cdot)$ to Λ as a group morphism.

The set of H-invariants of $\Gamma(X)$ is denoted by $\Gamma(X; \mathbf{1}_H) = \mathbb{C}[X]^H$. Then it decomposes multiplicity freely as a $\mathcal{G}_2$-module

$$\Gamma(X; \mathbf{1}_H) = \mathbb{C}[X]^H \simeq \sum_{\lambda \in \Lambda^+(H)}^{\oplus} \pi_2(\varphi(\lambda)).$$

Since $\Gamma(X; \mathbf{1}_H)$ is a finitely generated graded Noetherian algebra, it has a Poincaré series $P(\mathbf{1}_H; t)$, where t is an indeterminate. More precisely, we define $P(\mathbf{1}_H; t)$ in the following way. If the representation $\pi_1(\lambda) \boxtimes \pi_2(\varphi(\lambda))$ occurs in the k-th degree of the polynomial ring $\Gamma(X) = \mathbb{C}[X]$, we write $|\lambda| = k$. This degree map is obviously additive $|\lambda + \eta| = |\lambda| + |\eta|$. We put

$$P(\mathbf{1}_H; t) = \sum_{\lambda \in \Lambda^+(H)} \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|} = \operatorname{trace}_{\Gamma(X;\mathbf{1}_H)}(t^E), \tag{8.71}$$

where E denotes the degree operator. Let $\{a_1, \dots, a_d\} \subset \mathbb{C}[X]^H$ be a set of homogeneous and algebraically independent elements such that $\mathbb{C}[X]^H$ is integral over a subalgebra $\mathbb{C}[a_1, \dots, a_d]$ generated by $a_1, \dots a_d$. Put $h_i = \deg a_i$. Then there exists a polynomial $Q(t)$ such that

$$P(\mathbf{1}_H; t) = \frac{Q(t)}{\prod_{i=1}^d (1 - t^{h_i})}, \tag{8.72}$$

and $Q(1)$ gives a positive integer (see, e.g., [**45**, Theorem 2.5.6]). The integer $Q(1)$ is independent of the choice of $\{a_1, \dots, a_d\}$ above. We call it the *degree* of $\mathbb{C}[X]^H$ and denote $Q(1) = \operatorname{Deg}\Gamma(X; \mathbf{1}_H)$. The number d coincides with the transcendental degree of the quotient field of $\mathbb{C}[X]^H$, and we denote it by $d = \operatorname{Dim}\Gamma(X; \mathbf{1}_H)$, which is the dimension of the geometric quotient $X//H$.

More generally, for any $\sigma(\mu) \in \operatorname{Irr}(H)$ with highest weight μ, we denote $\sigma(\mu)$-covariants of $\Gamma(X)$ by $\Gamma(X; \sigma(\mu))$, i.e.,

$$\Gamma(X; \sigma(\mu)) := (\sigma(\mu)^* \otimes \mathbb{C}[X])^H.$$

The space of covariants $\Gamma(X; \sigma(\mu))$ is a finitely generated $\Gamma(X; \mathbf{1}_H) = \mathbb{C}[X]^H$-module by polynomial multiplication against the second factor (see, e.g., [**39**]). Note that it carries also a representation of $\mathcal{G}_2$ on the second factor.

If we decompose the restriction of $\pi_1(\lambda)$ to H as

$$\pi_1(\lambda)\big|_H \simeq \sum_{\mu}^{\oplus} m(\lambda, \mu)\, \sigma(\mu)$$

with multiplicity $m(\lambda, \mu)$, we have the decomposition

$$\Gamma(X; \sigma(\mu)) \simeq \sum_{\lambda \in \Lambda^+}^{\oplus} m(\lambda, \mu)\, \pi_2(\varphi(\lambda)),$$

as a $\mathcal{G}_2$-module. We define the Poincaré series $P(\sigma(\mu); t)$ of $\Gamma(X; \sigma(\mu))$ by

$$P(\sigma(\mu); t) = \sum_{\lambda \in \Lambda^+} m(\lambda, \mu) \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|}.$$

Since $\Gamma(X; \sigma(\mu))$ is a finitely generated graded module over $\Gamma(X; \mathbf{1}_H)$, its Poincaré series has rational expression as

$$P(\sigma(\mu); t) = \frac{Q(\sigma(\mu); t)}{\prod_{i=1}^d (1 - t^{h_i})}$$

with the same d and $h_1, \dots, h_d$ as in (8.72). Here, $Q(\sigma(\mu); t)$ is a polynomial in t and its value at $t = 1$ gives a non-negative integer, which is independent of the choice of $a_1, \dots, a_d$ again. We call it the *degree* of covariants $\Gamma(X; \sigma(\mu))$ and denotes $\operatorname{Deg}\Gamma(X; \sigma(\mu)) = Q(\sigma(\mu); 1)$.

The purpose of this subsection is to relate the dimension $\operatorname{Dim}\Gamma(X;\sigma(\mu))$ and the degree $\operatorname{Deg}\Gamma(X;\sigma(\mu))$ to those of invariants.

For "sufficiently large" λ, the multiplicity $m(\lambda,\mu)$ depends only on the coset $[\lambda] = \lambda+\Lambda(H) \in \Lambda^+/\Lambda(H)$. Here $\Lambda^+/\Lambda(H)$ is an abbreviation for $(\Lambda^+ + \Lambda(H))/\Lambda(H)$. To be precise, we have

***Lemma 8.1* (Sato).** — *For any $\lambda \in \Lambda^+$ and $\sigma(\mu) \in \operatorname{Irr}(H)$, there exists $\eta^M \in \Lambda^+(H)$ which satisfies*

$$m(\lambda+\eta^M,\mu) = m(\lambda+\eta^M+\eta,\mu) \qquad (\forall\eta \in \Lambda^+(H)).$$

The integer $m(\lambda+\eta^M,\mu)$ does not depend on the choice of η^M. We denote this integer by $m([\lambda],\mu)$ and call it the stable branching coefficient *after F. Sato.*

Proof. — Our setting here fits into Sato's assumption [**42**]. □

Let Δ_2^+ be a positive root system of $\mathcal{G}_2$. We define a subset $\Delta_2^+(H) \subset \Delta_2^+$ by

$$\Delta_2^+(H) = \{\alpha \in \Delta_2^+ \mid \langle\varphi(\eta),\alpha\rangle = 0 \quad (\forall\eta \in \Lambda(H))\}, \tag{8.73}$$

where $\langle,\rangle$ denotes the inner product which is invariant under the Weyl group action. For $\lambda \in \Lambda^+$, we put

$$r(\lambda) = r([\lambda]) = \prod_{\alpha\in\Delta_2^+(H)} \frac{\langle\varphi(\lambda)+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}, \tag{8.74}$$

where ρ is the half sum of positive roots in Δ_2^+. Note that the right hand side of (8.74) does not depend on individual λ, but depends only on the coset $[\lambda] \in \Lambda^+/\Lambda(H)$. By definition, $r([\lambda])$ is a positive quantity.

Proposition 8.2. — *We assume that, for any $\lambda \in \Lambda^+$, there exists $\lambda^\flat \in \Lambda^+$ such that*

$$(\lambda+\Lambda(H)) \cap \Lambda^+ = \lambda^\flat + \Lambda^+(H). \tag{8.75}$$

Then, for any $\mu \in \Lambda^+(H)$, we have

$$\lim_{t\uparrow 1} \frac{P(\sigma(\mu);t)}{P(\mathbf{1}_H;t)} = \operatorname{Deg}\Gamma(X;\mathbf{1}_H) \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu) r([\lambda]).$$

Remark 8.3. — Condition (8.75) determines $\lambda^\flat \in \Lambda^+$ uniquely if it exists. Hence, $\lambda^\flat$ depends only on the coset $[\lambda] = \lambda+\Lambda(H)$. If we set $S = \{\lambda^\flat \mid \lambda \in \Lambda^+\}$, this amounts to

$$\Lambda^+ = S \oplus \Lambda^+(H);$$

or, equivalently to say, Λ^+ is a free $\Lambda^+(H)$-module over the base set S. From this observation, the map

$$(\cdot)^\flat : \Lambda^+/\Lambda(H) \ni [\lambda] \mapsto [\lambda]^\flat := \lambda^\flat \in S \subset \Lambda^+$$

is a well-defined section of the projection map $\Lambda^+ \to \Lambda^+/\Lambda(H)$.

Corollary 8.4. — *Under the same assumption, we have*

$$\operatorname{Deg}\Gamma(X;\sigma(\mu)) \quad = \quad \operatorname{Deg}\Gamma(X;\mathbf{1}_H)\sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda]).$$

We need a technical lemma to prove the proposition.

Lemma 8.5. — *Take arbitrary* $\lambda\in\Lambda^+$.

(1) *There exists* $\eta_\lambda\in\Lambda^+(H)$ *such that*

$$\dim\pi_2(\varphi(\lambda+\eta))\le r(\lambda)\cdot\dim\pi_2(\varphi(\eta_\lambda+\eta)) \qquad (\forall\eta\in\Lambda^+(H)).$$

(2) *We have*

$$\dim\pi_2(\varphi(\lambda+\eta))\ge r(\lambda)\cdot\dim\pi_2(\varphi(\eta)) \qquad (\forall\eta\in\Lambda^+(H)).$$

Proof. — By Weyl's dimension formula, we have

$$\begin{aligned}\dim\pi_2(\varphi(\lambda+\eta)) &= \prod_{\alpha\in\Delta_2^+}\frac{\langle\varphi(\lambda+\eta)+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}\\ &= r(\lambda)\prod_{\alpha\notin\Delta_2^+(H)}\frac{\langle\varphi(\lambda+\eta)+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.\end{aligned}\tag{8.76}$$

To prove (1), it is enough to take $\eta_\lambda\in\Lambda^+(H)$ so that $\langle\varphi(\lambda),\alpha\rangle\le\langle\varphi(\eta_\lambda),\alpha\rangle$ holds for any $\alpha\notin\Delta_2^+(H)$. This is certainly possible. Since $\varphi(\cdot)$ is a group homomorphism, (8.76) becomes

$$r(\lambda)\prod_{\alpha\notin\Delta_2^+(H)}\frac{\langle\varphi(\lambda+\eta)+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}\le r(\lambda)\prod_{\alpha\notin\Delta_2^+(H)}\frac{\langle\varphi(\eta_\lambda+\eta)+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.$$

Now we are to prove (2). Since $\langle\varphi(\lambda),\alpha\rangle\ge 0$, we get

$$\dim\pi_2(\varphi(\lambda+\eta))\ge r(\lambda)\prod_{\alpha\notin\Delta_2^+(H)}\frac{\langle\varphi(\eta)+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}=r(\lambda)\dim\pi_2(\varphi(\eta)).$$

This proves (2). □

Proof of Proposition 8.2.. — Let us take arbitrary $0<t<1$.

First we note that $m(\lambda,\mu)\le m([\lambda],\mu)$ for any λ (see [**42**, Corollary 1.2]). Therefore, we have

$$\begin{aligned}P(\sigma(\mu);t) &= \sum_{\lambda\in\Lambda^+} m(\lambda,\mu)\dim\pi_2(\varphi(\lambda))\,t^{|\lambda|}\\ &\le \sum_{\lambda\in\Lambda^+} m([\lambda],\mu)\dim\pi_2(\varphi(\lambda))\,t^{|\lambda|}\\ &= \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)\sum_{\eta\in\Lambda^+(H)}\dim\pi_2(\varphi([\lambda]^\flat+\eta))\,t^{|[\lambda]^\flat+\eta|}.\end{aligned}\tag{8.77}$$

For $[\lambda]^\flat$, take $\eta_{[\lambda]^\flat} \in \Lambda^+(H)$ as in Lemma 8.5 (1), and recall the definition of $P(\mathbf{1}_H;t)$ from (8.71). Then we can calculate the above formula as

$$\begin{aligned}(8.77)\quad &\leq \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda])\ t^{|[\lambda]^\flat|-|\eta_{[\lambda]^\flat}|} \\ &\qquad\times \sum_{\eta\in\Lambda^+(H)} \dim \pi_2(\varphi(\eta_{[\lambda]^\flat}+\eta))\ t^{|\eta_{[\lambda]^\flat}+\eta|} \\ &\leq \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda])\ t^{|[\lambda]^\flat|-|\eta_{[\lambda]^\flat}|}\ P(\mathbf{1}_H;t).\end{aligned}$$

Note that, for fixed μ, there are only a finite number of cosets $[\lambda]$ for which $m([\lambda],\mu)$ does not vanish ([**42**, Corollary 2.5 (iii)]).

On the other hand, if we choose $\eta^M \in \Lambda^+(H)$ large enough, we have $m(\lambda+\eta^M,\mu) = m([\lambda],\mu)$ by the definition of the stable branching coefficient. We can take η^M uniformly for $\lambda \in \Lambda^+$, since there are only a finite number of $[\lambda]$'s which count. So, by Lemma 8.5 (2), we get the following inequality:

$$\begin{aligned}P(\sigma(\mu);t) &= \sum_{\lambda\in\Lambda^+} m(\lambda,\mu)\dim\pi_2(\varphi(\lambda))\ t^{|\lambda|} \\ &\geq \sum_{\lambda\in\Lambda^+} m(\lambda+\eta^M,\mu)\dim\pi_2(\varphi(\lambda+\eta^M))\ t^{|\lambda+\eta^M|} \\ &= \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu) \sum_{\eta\in\Lambda^+(H)} \dim\pi_2(\varphi([\lambda]^\flat+\eta^M+\eta))\ t^{|[\lambda]^\flat+\eta^M+\eta|} \\ &\geq \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda])\ t^{|[\lambda]^\flat+\eta^M|} \sum_{\eta\in\Lambda^+(H)} \dim\pi_2(\varphi(\eta))\ t^{|\eta|} \\ &= \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda])\ t^{|[\lambda]^\flat+\eta^M|}\ P(\mathbf{1}_H;t).\end{aligned}$$

From these inequalities, we have

$$\begin{aligned}&\sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda])\ t^{|[\lambda]^\flat|-|\eta_{[\lambda]^\flat}|} \\ &\qquad \geq \frac{P(\sigma(\mu);t)}{P(\mathbf{1}_H;t)} \geq \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda])\ t^{|[\lambda]^\flat+\eta^M|}.\end{aligned}$$

If we take the limit $t\uparrow 1$, we get

$$\lim_{t\uparrow 1}\frac{P(\sigma(\mu);t)}{P(\mathbf{1}_H;t)} = \sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu)r([\lambda]).$$

□

8.2. Examples of multiplicity free actions and Poincaré series. — We keep the notation in the former subsection §8.1. So $\mathcal{G}_1\times\mathcal{G}_2$ acts on X multiplicity freely, and H is a spherical subgroup of $\mathcal{G}_1$.

In many cases, we have an identity

$$\sum_{[\lambda]\in\Lambda^+/\Lambda(H)} m([\lambda],\mu) r([\lambda]) = \dim\sigma(\mu). \tag{8.78}$$

It will prove that

$$\operatorname{Deg}\Gamma(X;\sigma(\mu)) \quad = \quad \dim\sigma(\mu)\cdot\operatorname{Deg}\mathbb{C}[X]^H, \tag{8.79}$$

under the technical condition (8.75). However, at the same time, there also exist exceptions to (8.78). In this subsection, we will give three examples in which (8.78) and hence (8.79) hold. We need these examples later on.

Let B be a Borel subgroup of $\mathcal{G}_1$ such that $HB\subset\mathcal{G}_1$ is dense. Such a Borel subgroup exists since H is spherical. Define a parabolic subgroup $P\subset\mathcal{G}_1$ as

$$P=\{g\in\mathcal{G}_1 \mid HBg=HB\}\supset B.$$

Then $L=P\cap H$ is a reductive subgroup which contains the derived group of a Levi subgroup of P. The identity component of $B\cap H$ is a Borel subgroup of the identity component of L. Let $B=TU$ be a Levi decomposition with T being a Cartan subgroup of $\mathcal{G}_1$. We will denote by $\tau_L(\lambda)$ an irreducible representation of L with highest weight $e^\lambda\big|_{H\cap B}$.

Let Φ^+ be the semigroup lattice of dominant weights of $\mathcal{G}_1$ and Φ the weight lattice. We define

$$\Phi^+(H)=\{\lambda\in\Phi^+ \mid \dim V_\lambda^H=1\},$$

and denote by $\Phi(H)$ a lattice generated by $\Phi^+(H)$ in Φ. It is known that

$$\Phi(H)=\{\lambda\in\Phi \mid e^\lambda\big|_{H\cap T}\equiv 1\}.$$

To get the identity (8.78), we use *Sato's formula* ([**42**, Corollary 2.5])

$$\sum_{[\lambda]\in\Phi^+/\Phi(H)} m([\lambda],\mu)\dim\tau_L(\lambda) = \dim\sigma(\mu). \tag{8.80}$$

However, there are two obstructions to get identity (8.78) by using Sato's formula (8.80).

One obstruction is in the range of the summation. The representatives $[\lambda]$ must move all the coset of dominant weight lattice in Sato's formula. However, in general, $\Lambda^+/\Lambda(H)$ is a strict subset of $\Phi^+/\Phi(H)$. This obstruction is serious.

The other obstruction is the difference between $r(\lambda)$ and $\dim\tau_L(\lambda)$. However, in most cases, they are identical. We do not know an exception up to now.

We summarize here desired conditions which enables us to use Sato's formula.

(S1) Coincidence of coset spaces: $\Lambda^+/\Lambda(H)=\Phi^+/\Phi(H)$.

(S2) Coincidence of dimension functions: $r(\lambda)=\dim\tau_L(\lambda)\quad(\forall\lambda\in\Lambda^+)$.

(S3) Existence of good representatives: for any $\lambda\in\Lambda^+$, there exists $\lambda^\flat\in\Lambda^+$ such that

$$(\lambda+\Lambda(H))\cap\Lambda^+=\lambda^\flat+\Lambda^+(H). \tag{8.81}$$

Condition (S3) is equivalent to the following condition (S3′) (see Remark 8.3).

(S3′) There is a subset $S \subset \Lambda^+$ which satisfies $\Lambda^+ = S \oplus \Lambda^+(H)$.

If once we check the above conditions, we conclude the formula (8.79).

Theorem 8.6. — *If the above three conditions* (S1)–(S3) *hold, we have*

$$\operatorname{Deg}\Gamma(X;\sigma(\mu)) \;=\; \dim\sigma(\mu)\cdot\operatorname{Deg}\mathbb{C}[X]^H, \tag{8.82}$$

for any $\sigma(\mu)\in\operatorname{Irr}(H)$.

In the following, we examine the above three conditions (S1)–(S3) in each case.

Example A.

Let $\mathcal{G}_1 = GL(m,\mathbb{C}), \mathcal{G}_2 = GL(n,\mathbb{C})$ and assume that $m \le n$. This assumption is essential in the following. We take $H = SO(m,\mathbb{C})$. Therefore $(\mathcal{G}_1, H)$ is a symmetric pair. We put $X = M_{m,n}(\mathbb{C}) \simeq (\mathbb{C}^m \otimes \mathbb{C}^n)^*$ and let $\mathcal{G}_1 \times \mathcal{G}_2$ act naturally on X as

$$M_{m,n}(\mathbb{C}) \ni A \to {}^t g_1^{-1} A g_2^{-1}, \qquad (g_i \in \mathcal{G}_i, i = 1,2).$$

The decomposition of $\mathbb{C}[X]$ is given by

$$\mathbb{C}[X] \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} \tau_{GL_m}(\lambda)\boxtimes\tau_{GL_n}(\lambda),$$

where $\mathcal{P}_m$ denotes the set of partitions with length at most m. Therefore the action of $\mathcal{G}_1\times\mathcal{G}_2$ is multiplicity free, and we have

$$\Lambda^+ = \mathcal{P}_m, \qquad \Lambda = \Phi \simeq \mathbb{Z}^m.$$

Since we can naturally identify $\lambda\in\mathcal{P}_m$ with $\varphi(\lambda)\in\mathcal{P}_n$, we will denote $\varphi(\lambda)$ simply by the same letter λ. If we denote by $\mathcal{P}_m^{even}$ the set of even partitions, then it is well-known that

$$\Lambda^+(H) = \mathcal{P}_m^{even}, \qquad \Lambda(H) = \Phi(H) \simeq (2\mathbb{Z})^m.$$

In this case, the coset space $\Lambda^+/\Lambda(H) = \Lambda/\Lambda(H) \simeq (\mathbb{Z}_2)^m$ is a finite set, and it coincides with $\Phi^+/\Phi(H)$.

We have $\Delta_2^+(H) = \{\varepsilon_i - \varepsilon_j \mid m < i < j \le n\}$ in the standard notation. Using this, one can conclude easily that $r(\lambda) = 1$ for any $\lambda\in\mathcal{P}_m$. On the other hand, let B be a Borel subgroup consisting of upper triangular matrices. Then $HB \subset \mathcal{G}_1$ is dense and $P = B$. Since $L = H\cap P = H\cap B \simeq (\mathbb{Z}_2)^{m-1}$, L is a finite abelian group. Hence we have $r(\lambda) = \dim\tau_L(\lambda) = 1$.

Next we verify the condition (S3), i.e., (8.81). Put

$$\varpi_i = \sum_{k=1}^{i}\varepsilon_k = (1,\dots,1,0,\dots,0) \qquad (1\le i\le m),$$

the fundamental weights for $GL(m,\mathbb{C})$. Note that Λ^+ has a basis $\{\varpi_k \mid 1\le k\le m\}$ and $\Lambda^+(H)$ has a basis $\{2\varpi_k \mid 1\le k\le m\}$. Any $\lambda\in\Lambda^+$ can be expressed as

$$\lambda = \sum_{k=1}^{m} n_k\varpi_k \qquad (n_k\in\mathbb{Z}_{\ge 0}).$$

We put

$$ {n_k}^\flat = \begin{cases} 0 & \text{if } n_k \in 2\mathbb{Z}, \\ 1 & \text{otherwise} \end{cases} $$

Under the notation above, we define

$$ \lambda^\flat = \sum_{k=1}^{m} {n_k}^\flat \varpi_k. $$

It is a simple task to verify that $\lambda^\flat$ satisfies the condition (8.81), and we have

$$ \Lambda^+ = \Lambda^+(H) \oplus \left\{ \sum_{k=1}^{m} n_k \varpi_k \mid n_k = 0, 1 \right\}. $$

From observations above, we conclude Theorem 8.6 holds in this case.

Example B.
Let $\mathcal{G}_1 = GL(m, \mathbb{C}) \times GL(m, \mathbb{C})$, $\mathcal{G}_2 = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ and assume that $m \leq \min(p, q)$. We take $H = \Delta GL(m, \mathbb{C}) \simeq GL(m, \mathbb{C})$ the diagonal subgroup. Therefore $(\mathcal{G}_1, H)$ is a symmetric pair. Let $n = p + q$. We put

$$ X = M_{m,n}(\mathbb{C}) = M_{m,p}(\mathbb{C}) \oplus M_{m,q}(\mathbb{C}) \simeq (\mathbb{C}^m \otimes \mathbb{C}^p)^* \oplus (\mathbb{C}^m \otimes \mathbb{C}^q), $$

and let $((x_1, x_2), (y_1, y_2)) \in \mathcal{G}_1 \times \mathcal{G}_2$ act naturally on X as

$$ M_{m,n}(\mathbb{C}) \ni (A, B) \to ({}^t x_1^{-1} A y_1^{-1}, x_2 B \, {}^t y_2) \qquad (A \in M_{m,p}(\mathbb{C}), B \in M_{m,q}(\mathbb{C})). $$

Then the action is multiplicity free, and the decomposition of $\mathbb{C}[X]$ is given by

$$ \mathbb{C}[X] \simeq \sum_{(\mu,\nu) \in \mathcal{P}_m \times \mathcal{P}_m}^{\oplus} (\tau_{GL_m}(\mu) \boxtimes \tau_{GL_m}(\nu)^*) \boxtimes \left(\tau_{GL_p}(\mu) \boxtimes \tau_{GL_q}(\nu)^* \right). $$

Therefore we have

$$ \Lambda^+ = \mathcal{P}_m \times \mathcal{P}_m, \qquad \Lambda = \Phi \simeq \mathbb{Z}^m \times \mathbb{Z}^m. $$

Here, to avoid the confusion, we have twisted the second factor of Λ^+ by $-w_0$, where w_0 is the longest element in Weyl group. The correspondence between $\pi_1(\lambda)$ and $\pi_2(\varphi(\lambda))$ is given by

$$ \lambda = (\mu, \nu) \in \mathcal{P}_m \times \mathcal{P}_m \leftrightarrow \varphi(\lambda) = (\mu, \nu) \in \mathcal{P}_p \times \mathcal{P}_q $$

simply extended by zero. Again, we shall identify $\varphi(\lambda)$ with λ.

Since $\tau_{GL_m}(\mu) \boxtimes \tau_{GL_m}(\nu)^*$ contains non-trivial H-fixed vector if and only if $\mu = \nu$, we get

$$ \Lambda^+(H) = \Delta \mathcal{P}_m, \qquad \Lambda(H) = \Phi(H) \simeq \Delta \mathbb{Z}^m. $$

In this case, the coset space $\Lambda^+/\Lambda(H) = \Lambda/\Lambda(H) \simeq \mathbb{Z}^m$ is an infinite set, and it coincides with $\Phi^+/\Phi(H)$.

We have

$$ \Delta_2^+(H) = \{\varepsilon_i - \varepsilon_j \mid m < i < j \leq p\} \sqcup \{\delta_i - \delta_j \mid m < i < j \leq q\}, $$

in the standard notation, which concludes $r(\lambda) = 1$. We take a Borel subgroup $B = B_1 \times \overline{B_1} \subset \mathcal{G}_1$, where B_1 is the standard Borel subgroup of $GL(m, \mathbb{C})$ consisting of upper triangular matrices and $\overline{B_1}$ is its opposite. Then $HB \subset \mathcal{G}_1$ is dense. Again, the parabolic subgroup P coincides with B. Hence $L = H \cap P = H \cap B = \Delta T_1$ is isomorphic to a maximal torus T_1 in $GL(m, \mathbb{C})$. Therefore, we conclude that $r(\lambda) = 1 = \dim \tau_L(\lambda)$.

For $\lambda = (\mu, \nu) \in \Lambda^+$, let

$$\mu - \nu = \sum_{k=1}^{m} n_k \varpi_k \qquad (n_k \in \mathbb{Z}),$$

where $\{\varpi_k\}$ is the set of fundamental weights of $GL(m, \mathbb{C})$. Put

$$\mu^\flat = \sum_k \max(n_k, 0)\varpi_k, \qquad \nu^\flat = \sum_k \max(-n_k, 0)\varpi_k.$$

If we define $\lambda^\flat = (\mu^\flat, \nu^\flat)$, it satisfies the condition (8.81). In this case, we get

$$\Lambda^+ = \Lambda^+(H) \oplus \left\{ \left(\sum_{k=1}^{m} n_k \varpi_k, \sum_{k=1}^{m} n_k' \varpi_k \right) \mid n_k n_k' = 0,\ n_k, n_k' \in \mathbb{Z}_{\geq 0} \right\}.$$

Now we conclude that Theorem 8.6 also holds in this case.

Example C.

Let $\mathcal{G}_1 = GL(2m, \mathbb{C})$, $\mathcal{G}_2 = GL(p, \mathbb{C})$ and assume that $2m \leq p$. We take $H = Sp(2m, \mathbb{C})$. Therefore $(\mathcal{G}_1, H)$ is a symmetric pair. We realize $Sp(2m, \mathbb{C})$ as

$$Sp(2m, \mathbb{C}) = \left\{ g \in GL(2m, \mathbb{C}) \mid g \operatorname{diag}(J_2, \dots, J_2)\, {}^t g = \operatorname{diag}(J_2, \dots, J_2) \right\},$$

where

$$J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We put $X = M_{2m,p}(\mathbb{C}) \simeq (\mathbb{C}^m \otimes \mathbb{C}^p)^*$ and let $\mathcal{G}_1 \times \mathcal{G}_2$ act naturally on X as

$$M_{2m,p}(\mathbb{C}) \ni A \to {}^t g_1^{-1} A g_2^{-1}, \qquad (g_i \in \mathcal{G}_i, i = 1, 2).$$

The action of $\mathcal{G}_1 \times \mathcal{G}_2$ is multiplicity free, and we have the decomposition of $\mathbb{C}[X]$ as

$$\mathbb{C}[X] \simeq \sum_{\lambda \in \mathcal{P}_{2m}}^{\oplus} \tau_{GL_{2m}}(\lambda) \boxtimes \tau_{GL_p}(\lambda).$$

Therefore

$$\Lambda^+ = \mathcal{P}_{2m}, \qquad \Lambda = \Phi \simeq \mathbb{Z}^{2m}.$$

We shall identify $\lambda \in \mathcal{P}_{2m}$ with $\varphi(\lambda) \in \mathcal{P}_p$. The irreducible representation $\pi_1(\lambda)$ has a non-trivial H-fixed vector if and only if $\lambda_{2i-1} = \lambda_{2i}$ $(1 \leq i \leq m)$, i.e., $\lambda = \sum_{k=1}^{m} n_{2k} \varpi_{2k}$ $(n_{2k} \in \mathbb{Z}_{\geq 0})$. Therefore, we have

$$\Lambda^+(H) = \sum_{k=1}^{m} \mathbb{Z}_{\geq 0}\, \varpi_{2k}, \qquad \Lambda(H) = \Phi(H) = \sum_{k=1}^{m} \mathbb{Z}\, \varpi_{2k} \simeq \mathbb{Z}^m.$$

Then it is easy to see that

$$\Lambda^+/\Lambda(H) = \Phi^+/\Phi(H) \simeq \sum_{k=1}^{m} \mathbb{Z}_{\geq 0}\, \varpi_{2k-1}.$$

For $\lambda = \sum_{k=1}^{2m} n_k \varpi_k \in \Lambda^+$, we define

$$\lambda^\flat = \sum_{k=1}^{m} n_{2k-1}\varpi_{2k-1}.$$

Then it is clear that $\lambda^\flat$ satisfies the condition (8.81), hence we get

$$\Lambda^+ = \Lambda^+(H) \oplus \left\{ \sum_{k=1}^{m} n_k \varpi_{2k-1} \mid n_k \in \mathbb{Z}_{\geq 0} \right\}.$$

Take a Borel subgroup of $\mathcal{G}_1$ consisting of upper triangular matrices. Then $HB \subset \mathcal{G}_1$ is dense and the parabolic subgroup P is given by

$$P = \{\mathrm{diag}\,(p_1, p_2, \ldots , p_m) \mid p_k \in SL(2,\mathbb{C})\} \cdot B.$$

Then we have

$$L = H \cap P = \{\mathrm{diag}\,(p_1, p_2, \ldots , p_m) \mid p_k \in SL(2,\mathbb{C})\} \simeq SL(2,\mathbb{C})^m.$$

Therefore,

$$\tau_L(\lambda) = \tau_{SL_2}(\lambda_1 - \lambda_2) \boxtimes \tau_{SL_2}(\lambda_3 - \lambda_4) \boxtimes \cdots \boxtimes \tau_{SL_2}(\lambda_{2m-1} - \lambda_{2m}),$$

where $\tau_{SL_2}(\mu)$ is an irreducible representation of $SL(2,\mathbb{C})$ with highest weight μ. Since $\dim \tau_{SL_2}(\mu) = \mu + 1$, $\dim \tau_L(\lambda)$ is given by

$$\dim \tau_L(\lambda) = \prod_{k=1}^{m} (\lambda_{2k-1} - \lambda_{2k} + 1).$$

On the other hand, we have

$$\Delta_2^+(H) = \{\varepsilon_{2k-1} - \varepsilon_{2k} \mid 1 \leq k \leq m\} \sqcup \{\varepsilon_i - \varepsilon_j \mid 2m < i < j \leq p\}.$$

Hence we get

$$r(\lambda) = \prod_{k=1}^{m} (\lambda_{2k-1} - \lambda_{2k} + 1) = \dim \tau_L(\lambda).$$

Now we conclude that Theorem 8.6 is also valid in this case.

9. Associated cycle of unitary lowest weight modules

Let (G_1, G_2) be a reductive dual pair with G_2 being compact. We often write $G = G_1$ without subscription. We treat the three cases given in § 3 ; namely, $(G, G_2) = (Sp(2n, \mathbb{R}), O(m)), (U(p, q), U(m))$, or $(O^*(2p), Sp(2m))$.

In this section, we will prove the following theorem.

Theorem 9.1. — *We assume that the pair (G, G_2) is in the stable range where G_2 is a smaller member, i.e., $m \leq \mathbb{R}\text{-rank}\, G$.*

Take a finite dimensional irreducible representation $\sigma \in \mathrm{Irr}(G_2)$. Then the corresponding representation $L(\sigma) \in \mathrm{Irr}(\widetilde{G})$ is a unitary lowest weight module of the metaplectic cover $\widetilde{G}$ of G. The associated cycle of $L(\sigma)$ is given by

$$\mathcal{AC}\, L(\sigma) = \dim \sigma \cdot [\overline{\mathcal{O}_m}], \tag{9.83}$$

where $\mathcal{O}_m$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^-$ given in § 7.

Corollary 9.2. — *Let the notation be as above. Then, the Gelfand-Kirillov dimension and the Bernstein degree of $L(\sigma)$ are given by*

$$\mathrm{Dim}\, L(\sigma) = \dim \overline{\mathcal{O}_m}, \qquad \mathrm{Deg}\, L(\sigma) = \dim \sigma \cdot \deg \overline{\mathcal{O}_m}.$$

Explicit formulas for $\dim \overline{\mathcal{O}_m}$ and $\deg \overline{\mathcal{O}_m}$ are given in Theorems 7.5, 7.10 and 7.16.

Let us prove Theorem 9.1 for the pair $(Sp(2n, \mathbb{R}), O(m))$. This pair is the most complicated one, because $O(m)$ is not connected. The other pairs can be treated similarly.

Take an irreducible representation $\sigma \in \mathrm{Irr}(O(m))$ and consider the lowest weight module $L(\sigma)$ of $\widetilde{G} = \widetilde{Sp(2n, \mathbb{R})}$. First, let us recall the Poincaré series (6.43) of $L(\sigma)$

$$P(L(\sigma); t^2) = t^{-|\mu^+|} \sum_{\lambda \in \mathcal{P}_m} m(\lambda, \sigma)\, \dim \tau_\lambda^{(n)}\, t^{|\lambda|},$$

where $\tau_\lambda^{(n)}$ is an irreducible finite dimensional representation of $K_{\mathbb{C}} \simeq GL(n, \mathbb{C})$ with highest weight $\lambda \in \mathcal{P}_m$ and $\mathcal{P}_m$ is the set of all partitions of length less than or equal to m.

We consider two cases, according to $\sigma\big|_{SO(m)}$ is irreducible or not (see Lemma 6.1).

1) Let us assume that $\sigma\big|_{SO(m)}$ is irreducible. We denote by $\sigma(\mu) \in \mathrm{Irr}(SO(m))$ the restriction, where μ is the highest weight. In this case, the branching coefficient $m(\lambda, \sigma)$ satisfies

$$m(\lambda, \sigma) + m(\lambda, \sigma \otimes \det) = m(\lambda, \mu),$$

where $m(\lambda, \mu)$ is the branching coefficient with respect to $SO(m)$, i.e.,

$$\tau_\lambda^{(m)}\big|_{SO(m)} = \sum_{\mu}^{\oplus} m(\lambda, \mu) \sigma(\mu).$$

This means that

$$t^{|\mu|} P(L(\sigma); t^2) + t^{|\mu|+m-2k} P(L(\sigma \otimes \det); t^2) = \sum_{\lambda \in \mathcal{P}_m} m(\lambda, \mu) \dim \tau_\lambda^{(n)} t^{|\lambda|}, \tag{9.84}$$

where $\sigma = \sigma(\mu)$ (with the convention after Lemma 6.1) and $k = \ell(\mu)$. The right hand side of (9.84) coincides with the Poincaré series $P(\sigma(\mu);t)$ of covariants of $\sigma(\mu)$ defined in § 8, if we take $\mathcal{G}_1 = GL(m,\mathbb{C}) \supset H = SO(m,\mathbb{C}), \mathcal{G}_2 = GL(n,\mathbb{C})$, and $X = M_{m,n} = M_{n,m}^*$ as in Example A there. To distinguish two types of Poincaré series, we shall write $P(\Gamma(X;\sigma(\mu));t)$ instead of $P(\sigma(\mu);t)$ in this section.

Let $d = \operatorname{Dim} L(\mathbf{1}_{O(m)})$. Note that the Gelfand-Kirillov dimension of $L(\sigma)$ and $L(\sigma \otimes \det)$ also coincides with d. Then we have

$$\begin{aligned}\lim_{t\uparrow 1}(1-t^2)^d &\Big\{t^{|\mu|}P(L(\sigma);t^2) + t^{|\mu|+m-2k}P(L(\sigma\otimes\det);t^2)\Big\}\\ &= \operatorname{Deg} L(\sigma) + \operatorname{Deg} L(\sigma\otimes\det).\end{aligned} \tag{9.85}$$

This implies that $d = \operatorname{Dim}\Gamma(X;\sigma(\mu))$ and

$$\lim_{t\uparrow 1}(1-t^2)^d P(\Gamma(X;\sigma(\mu));t) = \operatorname{Deg}\Gamma(X;\sigma(\mu)). \tag{9.86}$$

Lemma 9.3. — *For any $\sigma \in \operatorname{Irr}(O(m))$, we have*

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg} L(\sigma\otimes\det).$$

Proof. — We denote a subspace of the symmetric algebra $S(M_{n,m}) = \mathbb{C}[M_{n,m}^*]$ on which $O(m)$ acts via σ by V_σ. Then the representation space of $L(\sigma)$ is identified with the σ-covariants $(V_\sigma \otimes \sigma^*)^{O(m)}$. In order to get the $\widetilde{K}$-action on it, we must twist it by $(\det k)^{m/2}$ $(k \in GL(n,\mathbb{C}))$, though it does not affect on the gradation itself. Since we only consider the Poincaré series, we simply ignore this twist.

Put

$$\delta = \det\big(E_{ij}\big)_{1\le i,j\le m} \in S(M_{n,m}),$$

where E_{ij} is the matrix unit. Then, clearly δ represents $\det \in \operatorname{Irr}(O(m))$. The multiplication by δ maps V_σ injectively to $V_{\sigma\otimes\det}$,

$$\delta : V_\sigma \longrightarrow V_{\sigma\otimes\det}.$$

This map increases the degree by $\deg\delta = m^2$, and we conclude that

$$t^{m^2}P(L(\sigma);t) \le P(L(\sigma\otimes\det);t)$$

for $0 < t < 1$. Since $(\sigma\otimes\det)\otimes\det = \sigma$, we finally get

$$t^{2m^2}P(L(\sigma);t) \le t^{m^2}P(L(\sigma\otimes\det);t) \le P(L(\sigma);t).$$

If we multiply $(1-t)^d$ $(d = \operatorname{Dim} L(\sigma))$ and take limit $t \uparrow 1$, we get

$$\operatorname{Deg} L(\sigma) \le \operatorname{Deg} L(\sigma\otimes\det) \le \operatorname{Deg} L(\sigma).$$

$\square$

By Lemma 9.3, formulas (9.3) and (9.86) imply

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg} L(\sigma\otimes\det) = 2^{-1}\operatorname{Deg}\Gamma(X;\sigma(\mu)). \tag{9.87}$$

Consider a special case where $\sigma = \mathbf{1}_{O(m)}$, the trivial representation of $O(m)$. Then the above formula (9.87) becomes

$$\operatorname{Deg} L(\mathbf{1}_{O(m)}) = \operatorname{Deg} L(\det) = 2^{-1}\operatorname{Deg}\Gamma(X;\mathbf{1}_{SO(m)}). \tag{9.88}$$

Theorem 8.6 implies that

$$\begin{aligned}\operatorname{Deg} L(\sigma) &= 2^{-1}\operatorname{Deg}\Gamma(X;\sigma(\mu))\\ &= 2^{-1}\dim\sigma(\mu)\ \operatorname{Deg}\Gamma(X;\mathbf{1}_{SO(m)}) = \dim\sigma(\mu)\ \operatorname{Deg} L(\mathbf{1}_{O(m)}).\end{aligned}$$

Since the associated cycle of $L(\sigma)$ is a multiple of $\overline{\mathcal{O}_m}$, the multiplicity is given by

$$\operatorname{Deg} L(\sigma)/\deg\overline{\mathcal{O}_m} = \operatorname{Deg} L(\sigma)/\operatorname{Deg} L(\mathbf{1}_{O(m)}) = \dim\sigma(\mu) = \dim\sigma$$

(cf. Theorems 1.4 and 7.5).

2) Assume that $\sigma\big|_{SO(m)} = \sigma(\mu^+)\oplus\sigma(\mu^-)$ as in Lemma 6.1 (2). Then it is easy to see that

$$m(\lambda,\sigma) = m(\lambda,\mu^+) = m(\lambda,\mu^-).$$

Therefore we have

$$t^{|\mu^+|}P(L(\sigma);t^2) = \sum_{\lambda\in\mathcal{P}_m} m(\lambda,\mu^+)\dim\tau_\lambda^{(n)}t^{|\lambda|} = P(\Gamma(X;\sigma(\mu^+));t).$$

Multiply $(1-t^2)^d$ both hand sides, and take limit $t\uparrow 1$. Then we get

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg}\Gamma(X;\sigma(\mu^+)) = \dim\sigma(\mu^+)\ \operatorname{Deg}\Gamma(X;\mathbf{1}_{SO(m)}).$$

By (9.88), we get

$$\operatorname{Deg} L(\sigma) = 2\dim\sigma(\mu^+)\ \operatorname{Deg} L(\mathbf{1}_{O(m)}) = \dim\sigma\ \operatorname{Deg} L(\mathbf{1}_{O(m)}),$$

which proves (9.83) by the same reasoning as 1). This completes the proof of Theorem 9.1 for the pair $(Sp(2n,\mathbb{R}),O(m))$.

For the other pairs, we use Examples B and C in § 8 instead of Example A. In these cases, we have

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg}\Gamma(X;\sigma)$$

for appropriate choice of X. This formula and Theorem 8.6 prove the theorem by almost the same arguments above.

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